Probability and Statistics
for Final Year Engineering Students

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Home Work Project #1

Due date: Tuesday May 24

Hand in instructions: Please hand in a stapled hard-copy no later than the due date. Your hand in may be in a combination of handwriting and computer output. Make sure all your graphs are clear, well-scaled, and properly labeled. A portion of the grade will be based on the clarity of the presentation.

1) **Normal approximation to the binomial distribution.**
Let $B_1, B_2, \dots$ be a sequence of binomial random variables with B_n having a number of trials parameter equal to n and success probability parameter p (the same value for all random variables in the sequence). Let μ_n be the sequence of means, $E[B_n] = \mu_n$ and let σ_n^2 be the sequence of variances, $Var(B_n) = \sigma_n^2$. Denote $\tilde{B}_n = \frac{B_n - \mu_n}{\sigma_n}$.
    a. For $p = 1/3$ and $p = 1/2$, plot the sequence of PDF's of $\tilde{B}_n$ for n=1,2,3,4,5,10,20,30,40,50,100.
    b. Let $q_n = P(B_n \geq \mu_n + 2\,\sigma_n)$. For $p = 1/3$ and $p = 1/2$, calculate q_n for n=1,…,100, put the results in a table.
    c. Use a normal (CLT) approximation to approximate the results of (b). Put the results in a table, showing the errors.

2) **Analytical investigation of the exponential distribution.**
X is an exponential random variable with parameter $\lambda > 0$ if the density is $f(x) = \lambda e^{-\lambda x}$ for $x \geq 0$ and 0 elsewhere.
    a. Find the CDF of X and show your calculation.
    b. Calculate the mean of X.
    c. Calculate the variance of X.
    d. Let $M_k = E[X^k]$. (This is called the k'th moment of X). What is M_0? Find a recursive formula for M_k in terms of M_{k-1}.

3) Properties of variance estimators.

Let $X_1, \dots, X_n$ be a sequence of independent uniform [0,1] random variables.

a. Calculate $\sigma^2 = Var(X_i)$.

b. Estimate σ^2 using the sample variance. Plot your estimate as an increasing function of the sample size, n.

c. Fix n=5. Use simulation to plot the distribution of the sample variance. What is the mean of this distribution?

d. Let $\tilde{S}^2 = \frac{\sum_{i=1}^{n}(X_i - 1/2)^2}{n}$. Is this an unbiased estimator of the variance? Prove your result. Show your result by means of simulation (use n =5).

4) The Gamma distribution.

X is a Gamma random variable with parameters $\lambda > 0$ and $\alpha > 0$, if the density is $f(x) = C\, x^{\alpha-1}\, e^{-\lambda x}$ for $x \geq 0$ and some $C > 0$.

a. Find an expression for the normalizing constant C. You may use the “gamma function” (Google it). Make sure you define the gamma function in your result.

b. Plot gamma densities for different parameter values.

c. Is the exponential distribution a special case?

d. Find the mean and variance of the gamma distribution – show your calculations.

e. Assume now that $X_1, \dots, X_n$ is a random sample from a gamma distribution with unknown parameters λ and α. One way to estimate the parameters is the **method of moments**. In this method, you equate the sample mean and sample variance to the mean and variance expressions and solve for λ and α. Write the equations for the method of moments and then write the resulting estimators (functions of the random sample), $\hat{\lambda}$ and $\hat{\alpha}$.

f. Use simulation to check if the estimators in (d) are biased/asymptotically unbiased – show your results for three arbitrary pairs of values of the parameters. Note: We have not discussed how to generate Gamma random variables – you can look this up or read the help of the mathematical package you are using.

Good Luck.