Probability and Statistics
for Final Year Engineering Students

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Lecture 5p:
Basics of Stochastic Processes: Definitions, Examples, Stationary, Mean, Autocorrelation

Stochastic = Random.

A stochastic process is a collection of random variables indexed by a time parameter. E.g. $\{X_n, n = 1,2,\dots\}$ discrete time or $\{X_t, t \geq 0\}$ continuous time.

Before we looked at a random sample, assuming that the random variables are **i.i.d. (independent and identically distributed)** – this is a "pretty boring" stochastic process.

Examples of phenomena that can be modeled by stochastic processes:

- **Weather.**
- **Congestion.**
- **Communication.**
- **Images.**
- **Biology**
- **….**

The **distribution of the stochastic process** is the specification of

P$(X_{t_1} = x_1, \dots, X_{t_k} = x_k)$ for all k and $t_1, \dots, t_k$. In particular, the **marginal distribution at time t** is the specification of $P(X_t = i)$ (or density if continuous space).

The **mean function** of the stochastic process is:

$$m(t) = E[X_t].$$

The **variance function** is:

$$V(t) = Var(X_t).$$

The **autocovariance** is,

$$C(t_1, t_2) = Cov(X_{t_1}, X_{t_2})$$

We will often be interested about processes where $(t) = 0$ $(zero\ mean)$, for these the,

Autocorrelation is useful:

$$R(t_1,t_2) = E\left[X_{t_1}X_{t_2}\right] = in\ case\ zero\ mean = \ C(t_1,t_2).$$

A "Fully Worked" Example, a binomial counting process:

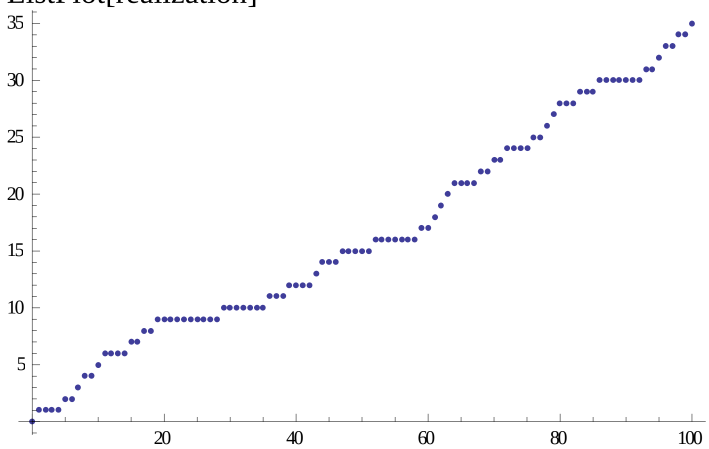
Let $I_1, I_2, \ldots$ be an infinite sequence of i.i.d. binary random variables with $P(I_i = 1) = p$.

Denote $X_0 = 0$ and $X_n = \sum_{i=1}^{n} I_i = X_{n-1} + I_n$.

Simulating a single trajectory sometimes called **realization** or **sample path** of X_n is not hard:

```
p=0.3;
binaryRV:=If[Random[]≤ p, 1,0]
NN=100;
binarySequence=Table[binaryRV,{100}];
xSequence[n_]:=If[n==0,0,Total[binarySequence[[1;;n]]]]
realization=Table[{n,xSequence[n]},{n,0,NN}];
```

```
ListPlot[realization]
```



For such a stochastic process let us analyze:

- The marginal distribution at time t (Binomial(t,p))
- The mean function (m(t)=t p)
- The variance function (V(t) = t p(1-p))
- The joint distribution at two times, t_1 and t_2.
- The autocovariance.
- We can in essence quite easily get the full distribution of the process (i.e. for any finite sequence of times, we can calculate the joint distribution).

Gaussian Random Processes:

A random process X(t) is a **Gaussian random process if the joint distribution of** $X_{t_1}, \dots, X_{t_k}$ for for all k and $t_1, \dots, t_k$ is jointly Gaussian.

We saw the bivariate (two – dimensional) joint Gaussian distribution in the previous section. This can be generalized to an arbitrary finite dimension.

The **mean vector** of a vector of random variables, X=($X_1, \dots, X_n)'$ is:

$m_X = \begin{bmatrix} E[X_1] \\ \vdots \\ E[X_n] \end{bmatrix}$. The covariance matrix, K_X is the n by n symmetric matrix with element i,j being $Cov(X_i, X_j)$.

Exercise: For the bivariate normal, write the mean and covariance matrix.

The distribution of a **jointly Gaussian random vector** is fully described by the mean and covariance matrix. The density is as follows:

$$f(\boldsymbol{x}) = f(x_1, \dots, x_n) = \frac{1}{(2\pi)^{n/2}\det(K_X)^{1/2}} \exp\{-\frac{1}{2}(\boldsymbol{x} - m_X)' {K_X}^{-1}(\boldsymbol{x} - m_X)\}.$$

Exercise: Show that the single variable and bivariate normal are special cases.

Observe: Since a joint Gaussian distribution is fully characterized by the mean and covariance matrix, we have that the distribution of Gaussian random process is fully determined by the functions m(t) and $C(t_1, t_2)$**.** Why?

Stationary Random Processes:

A random process is stationary if the distribution of $X_{t_1}, \dots, X_{t_k}$ is the same as $X_{t_1+\tau}, \dots, X_{t_k+\tau}$ is the same for all τ. This is somewhat similar to a system (in system theory) being "time-invariant".

Wide-Sense Stationary Random Processes:

A random process is **wide-sense stationary (WSS)** if $m(t) = m$ and $C(t_1, t_2)$ only depends on the difference of the times t_2 and t_1, i.e. $C(t_1, t_2) = C(\tau = t_2 - t_1)$**.** Or alternatively $R(t_1, t_2) = R(\tau = t_2 - t_1)$.

Any stationary random process is WSS. Why?

Any WSS Gaussian random process is stationary. Why?

From now on, we will concentrate on WSS Gaussian processes (typically with zero mean). The distribution of such processes is fully described by their Autocorrelation function $R(\tau)$.

In signal processing, control and analysis of stochastic systems it is often useful to model the signals as being WSS random processes (and even) Gaussian. After "renormalization" (subtracting by the mean) the signals are zero mean and thus $R(\tau)$ describes their behavior.

Properties of the autocorrelation function:

- $R(0) = E[{X_t}^2]$ for all t. This is often called the **average power** of the signal. (Why power?: If one views the signal as being voltage than voltage*current = power but current is proportional to voltage so power is proportional to voltage squared.)
- $R(\tau)$ is an even function:

$$R(\tau) = R(-\tau)$$

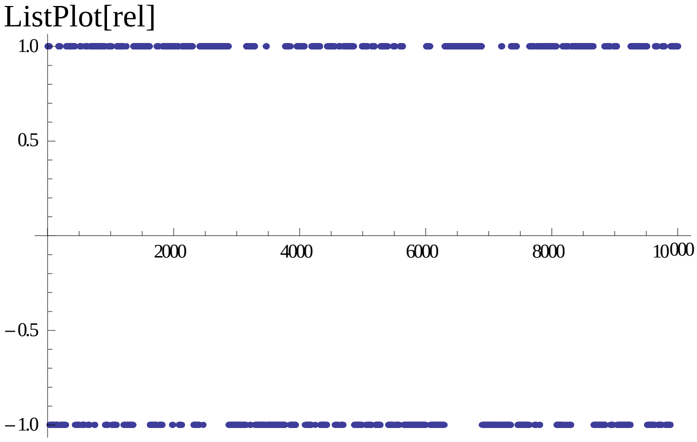
- $R(\tau)$ gets a maximum at $\tau = 0$.
- $P(|X(t+\tau) - X(t)| > \varepsilon) \leq 2\frac{R(0)-R(\tau)}{\varepsilon^2}$.

  This shows that if $R(\tau)$ "drops off slowly" when moving away from 0, then the probabilities of large changes in τ time units is small.

- If $R(0) = R(d)$ then R() is periodic with period d.

Rough illustration of estimation of the correlation of a telegraph type process:

```
p1=0.01;
p2=0.01;
next[x_]:=If[x==1,If[Random[]≤ p1, -1, 1],
  If[Random[]≤ p2, 1, -1]
  ]
NN=10000;
rel=NestList[next,1,NN];
ListPlot[rel]
```




```
corrSample[k_]:=Mean[Table[ rel[[i]] rel[[i+k]],{i, 1,NN-k}] //N]
```

```
corrEst=Table[{k,corrSample[k]},{k,0,100}];
ListPlot[corrEst,PlotRange→All]
```