# Probability and Statistics
# for Final Year Engineering Students

By Yoni Nazarathy, Last Updated: May 24, 2011.

## Lecture 6p:
## Spectral Density, Passing Random Processes through LTI Systems, Filtering

**Terms and concepts from previous lecture:**

The **mean function**: $m(t) = E[X_t]$.

The **variance function**: $V(t) = Var(X_t)$.

The **autocovariance**: $C(t_1, t_2) = Cov(X_{t_1}, X_{t_2})$

The **Autocorrelation** function: $R(t_1, t_2) = E\left[X_{t_1} X_{t_2}\right] = in\ case\ zero\ mean = \ C(t_1, t_2)$.

A random process X(t) is a **Gaussian random process if the joint distribution of** $X_{t_1}, \dots, X_{t_k}$ for for all k and $t_1, \dots, t_k$ is jointly Gaussian. This is the density of a joint Gaussian distribution (with mean vector $m_X$ and covariance matrix $K_X$):

$$f(\boldsymbol{x}) = f(x_1, \dots, x_n) = \frac{1}{(2\pi)^{n/2} \det(K_X)^{1/2}} \exp\{-\frac{1}{2}(\boldsymbol{x} - m_X)' {K_X}^{-1} (\boldsymbol{x} - m_X)\ \}.$$

Since a joint Gaussian distribution is fully characterized by the mean and covariance matrix, we have that the distribution of Gaussian random process is fully determined by the functions m(t) and $C(t_1, t_2)$.

A random process is **stationary** if the distribution of $X_{t_1}, \dots, X_{t_k}$ is the same as $X_{t_1+\tau}, \dots, X_{t_k+\tau}$ is the same for all $\tau$.

**Wide-Sense Stationary Random Processes:**

A random process is **wide-sense stationary (WSS)** if $m(t) = m$ and $C(t_1, t_2)$ only depends on the difference of the times $t_2$ and $t_1$, i.e. $C(t_1, t_2) = C(\tau = t_2 - t_1)$. Or alternatively $R(t_1, t_2) = R(\tau = t_2 - t_1)$.

Any stationary random process is WSS (the reverse does not always hold).

Any WSS Gaussian random process is stationary.

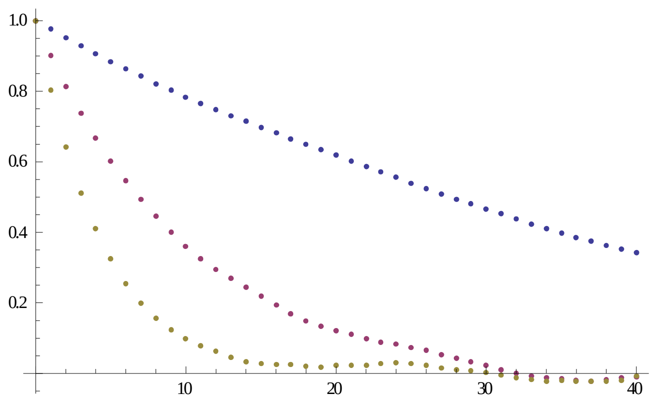
From now on, we will concentrate on WSS Gaussian processes (typically with zero mean). The distribution of such processes is fully described by their Autocorrelation function $R(\tau)$.

**Properties of the autocorrelation function:**

- $R(0) = E[{X_t}^2]$ for all t.
- $R(\tau)$ is an even function: $R(\tau) = R(-\tau)$
- $R(\tau)$ gets a maximum at $\tau = 0$.
- $P(|X(t+\tau) - X(t)| > \varepsilon) \leq 2\frac{R(0)-R(\tau)}{\varepsilon^2}$.
- If $R(0) = R(d)$ then R() is periodic with period d.

Rough illustration of estimation of the correlation of a telegraph type process:

```
NN=10000;
p1=0.01;p2=0.01;
next[x_]:=If[x==1,If[Random[]≤p1,-1,1],If[Random[]≤p2,1,-1]]
rel1=NestList[next,1,NN];
corrSample[k_]:=Mean[Table[rel1[[i]] rel1[[i+k]],{i,1,NN-k}]//N]
corrEst1=Table[{k,corrSample[k]},{k,0,40}];
p1=0.05;p2=0.05;
next[x_]:=If[x==1,If[Random[]≤p1,-1,1],If[Random[]≤p2,1,-1]]
rel2=NestList[next,1,NN];
corrSample[k_]:=Mean[Table[rel2[[i]] rel2[[i+k]],{i,1,NN-k}]//N]
corrEst2=Table[{k,corrSample[k]},{k,0,40}];
p1=0.1;p2=0.1;
next[x_]:=If[x==1,If[Random[]≤p1,-1,1],If[Random[]≤p2,1,-1]]
rel3=NestList[next,1,NN];
corrSample[k_]:=Mean[Table[rel3[[i]] rel3[[i+k]],{i,1,NN-k}]//N]
corrEst3=Table[{k,corrSample[k]},{k,0,40}];
ListPlot[{corrEst1,corrEst2,corrEst3},PlotRange→All,AxesOrigin→{0,0}]
```



**The power spectral density:**

The **power spectral density** is the Fourier transform of the autocorrelation function.

$$S(f) = \int_{-\infty}^{\infty} R(\tau) e^{-i2\pi f\tau} d\tau$$

Since the autocorrelation function is an even function:

$$S(f) = \int_{-\infty}^{\infty} R(\tau)(cos\ 2\pi f\tau - j\ sin\ 2\pi f\tau) d\tau = \int_{-\infty}^{\infty} R(\tau) cos\ 2\pi f\tau\ d\tau.$$

So $S(f)$ is a real valued even function.

Given a random trajectory of the process over a finite interval [0,T] we can compute the Fourier transform of this trajectory:

$$\tilde{x}(f) = \int_{0}^{T} X(t)\, e^{-i2\pi ft} dt$$

We can then approximate the "power density" as a function of frequency by,

$$\tilde{p}_T(f) = \frac{1}{T}|\tilde{x}(f)|^2 = \frac{1}{T}\tilde{x}(f)\tilde{x}^*(f) = \frac{1}{T}\left(\int_0^T X(t)\, e^{-i2\pi ft} dt\right)\left(\int_0^T X(t)\, e^{i2\pi ft} dt\right).$$

(This is what a spectrum analyzer does),

It then turns out (by the (Einstein)-Wiener-Khinchin theorem) that,

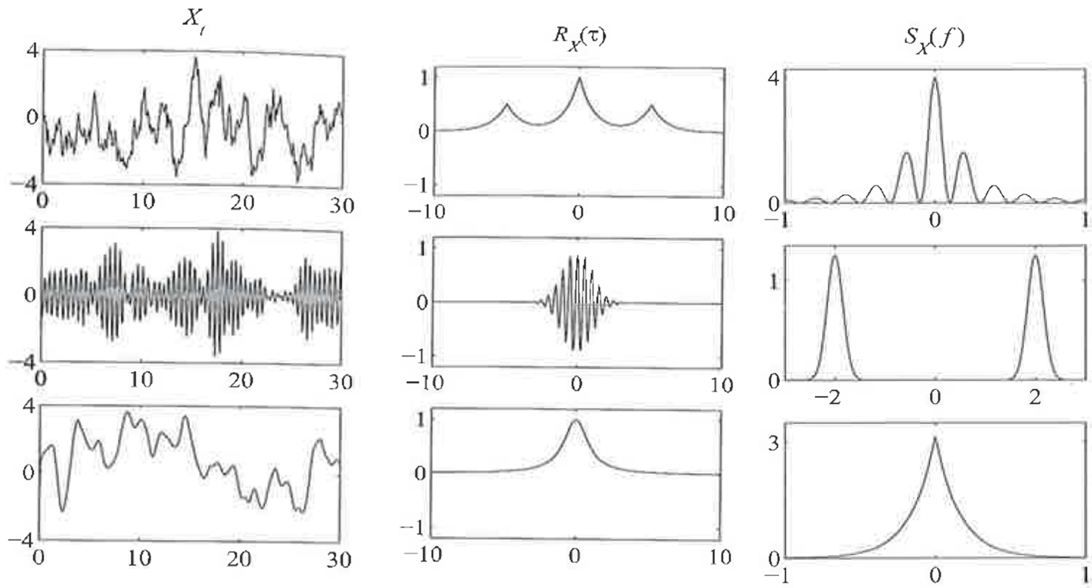
$$S(f) = \lim_{T\to\infty} E[\tilde{p}_T(f)].$$

Given the power spectral density we can get the autocorrelation function by the inverse Fourier transform:

$$R(\tau) = \int_{-\infty}^{\infty} S(f) e^{i2\pi f\tau} df.$$

Note that the **average power** of the process is,

$$E[X(t)^2] = R(0) = \int_{-\infty}^{-\infty} S(f) df.$$




**Random Processes Passing Through Linear Time Invariant (LTI) Systems:**

A Linear Time Invariant (LTI) system transforms an input signal $X_t$ into an output signal $Y_t$ in a way such that the output of a linear combination of signals is a linear combination of the outputs and the output of a time shifted signal is the time shifted output. Many physical and signal processing systems can be modeled and implemented as LTI.

A useful property of LTI systems is that the impulse response $h(t)$ fully describes the system in that the output resulting from any input signal is the convolution of the input with the impulse response.

$$Y_t = \int_{-\infty}^{\infty} h(t-u)X_u du$$

We now consider random signals passing through LTI systems.

The mean and autocorrelation of the output are as follows:

$$m_Y(t) = \int_{-\infty}^{\infty} h(t-u)m_X(u)du.$$

$$R_Y(\tau) = \int_{-\infty}^{\infty} h(u)\left(\int_{-\infty}^{\infty} h(s)R(\tau-u+s)ds\right)du.$$

The transfer function of an LTI is,

$$H(f) = \int_{-\infty}^{\infty} h(\tau)e^{-i2\pi f\tau}d\tau.$$

KEY Result: for WSS processes: $S_Y(f) = |H(f)|^2 S_X(f)$ .

The above result shows us how we can use linear systems to filter the power spectral density of random signals: Pass the signal through a linear system with a desired transfer function.

**Example:** White noise through a low pass RC filter.

A WSS process is called **white noise** if $R(\tau) = c\delta(\tau)$. Alternatively $S(f) = c$.

Consider an RC circuit. Suppose the voltage source is a white noise process and the output is the capacitor voltage. We find the power spectral density of the output process.

The transfer function of this system is $H(f) = \frac{1}{1+i2\pi RC\, f}$.

So, $S_Y(f) = |H(f)|^2 S_X(f) = \frac{1}{1+i2\pi RC\, f}\left(\frac{1}{1+i2\pi fRC}\right)^* c = \frac{1}{1+(2\pi fRC)^2} c.$

**Example:**

White noise is applied to a filter with impulse response, $h(t) = I_{[0,T]}(t)$.

$$H(f) = \int_{-\infty}^{\infty} h(t)e^{-i2\pi ft}dt = \frac{1-e^{-i2\pi fT}}{i2\pi f} = e^{-i\pi Tf}T\frac{\sin(\pi\, Tf)}{\pi\, Tf}.$$

$$\text{So,} \qquad S_Y(f) = |H(f)|^2 S_X(f) = T^2\left(\frac{\sin(\pi\, Tf)}{\pi\, Tf}\right)^2 c.$$

**Design of a matched filter:**

Assume we may be receiving a known signal $v(t)$ in and have additive noise (not necessarily white) $X_t$. Examples are in detection of communication messages and in radars.

In case there is in fact a known signal, we receive $v(t) + X_t$. We can filter this input through an LTI to obtain $v_0(t) + X_t$.

An important design question is in choosing the LTI that will maximize the signal to noise ratio at time $t_0$:

$$SNR = \frac{v_0(t_0)^2}{E[{Y_{t_0}}^2]} = \frac{v_0(t_0)^2}{P_Y},$$

Where $P_Y = R_Y(0)$.

It turns out that the best filter to use is,

$$H(f) = \frac{1}{S_X(f)}V(f)^* e^{-i2\pi ft_0}.$$

Observe that the inverse Fourier transform of $V(f)^* e^{-i2\pi ft_0}$ is $v(t_0 - t)$.

So in case the noise is white the best filter to use a system with impulse response $v(t_0 - t)$.

**Derivation of the matched filter:**

$$P_Y = \int_{-\infty}^{\infty} |H(f)|^2 S_X(f)\, df$$

$$v_0(t_0) = \int_{-\infty}^{\infty} H(f) V(f) e^{i2\pi f t_0}\, df$$

So,

$$v_0(t_0) = \int_{-\infty}^{\infty} H(f)\sqrt{S_X(f)}\,\frac{V(f)e^{i2\pi f t_0}}{\sqrt{S_X(f)}}\, df = \int_{-\infty}^{\infty} H(f)\sqrt{S_X(f)}\left|\frac{V(f)^* e^{-i2\pi f t_0}}{\sqrt{S_X(f)}}\right|^* df$$

Now we use this version of the Cauchy-Schwarz inequality:

$\left|\int_{-\infty}^{\infty} g(\theta)h(\theta)^* d\theta\right|^2 \le \int_{-\infty}^{\infty} |g(\theta)|^2 d\theta \cdot \int_{-\infty}^{\infty} |h(\theta)|^2 d\theta.$

and equality holds only if h() is a multiple of g().

So,

$$|v_0(t_0)|^2 \le \int_{-\infty}^{\infty} |H(f)|^2 S_X(f)\, df \cdot \int_{-\infty}^{\infty} \frac{|V(f)|^2}{S_X(f)}\, df = P_Y\, B.$$

So,

$$SNR = \frac{|v_0(t_0)|^2}{P_Y} \le \frac{P_Y\, B}{P_Y} = B.$$

This upper bound is attained if the inequality is an equality which occurs when $H(f)\sqrt{S_X(f)}$ is a multiple of $\frac{V(f)^* e^{-i2\pi f t_0}}{\sqrt{S_X(f)}}$:

$$H(f)\sqrt{S_X(f)} = \alpha \frac{V(f)^* e^{-i2\pi f t_0}}{\sqrt{S_X(f)}}.$$

This, after re-organizing (and choosing $\alpha = 1$) gives the desired frequency response of the system:

$$H(f) = \frac{1}{S_X(f)} V(f)^* e^{-i2\pi f t_0}.$$