

Probability and Statistics for Final Year Engineering Students

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Exercises and Tutorial 0: Preliminaries and Mathematical Review

Basic Summations:

The notation $\sum_{i=1}^n a_i$ implies $a_1 + a_2 + \dots + a_n$. There are many straight forward manipulations:

- $\sum_{i=1}^n c = nc$
- $\sum_{i=1}^n ca_i = c \sum_{i=1}^n a_i$
- $\sum_{i=k}^n a_i = \sum_{i=0}^{n-k} a_{i+k}$
- $\sum_{i=1}^n (a_i + b_i) = \sum_{i=1}^n a_i + \sum_{i=1}^n b_i$

In certain cases where the sequence a_i is structured, the sums can be summarized by specific formulas:

- $\sum_{i=1}^n i = \frac{n(n+1)}{2}$
- $\sum_{i=1}^n i^2 = \frac{n(n+1)(2n+1)}{6}$
- $\sum_{i=1}^n q^i = \frac{q-q^{n+1}}{1-q} \quad q \neq 1$

- 1) Verify the above 3 formulas for $n=1,2,3,4$.
- 2) Calculate: $\sum_{i=1}^{100} (5i + 2)$
- 3) Calculate: $\sum_{i=10}^{30} (2i + 4i^2)$
- 4) Calculate: $\sum_{i=0}^{n-1} 2^i$

Notation and basic properties of sets:

- $A = \{a_1, a_2, \dots, a_n\}$ is a set of elements.
- $|A|$ denotes the number of points in the set when $n < \infty$, otherwise there can be an infinite number of points in the set (either countable infinity or uncountable infinity).
- Denote $x \in A$ when the element x is in A , otherwise denote $x \notin A$.
- The union of two sets, $A \cup B$ is the set of elements that are contained in A or B or both, e.g.: $\{4, 1, x\} \cup \{1, 9, -2\} = \{4, 1, 9, -2, x\}$.
- Note: sets are typically regarded as unordered.
- The set A is contained in the set B , $A \subseteq B$ if for every $x \in A$, $x \in B$.
- The sets are equal ($A=B$), if both $A \subseteq B$ and $B \subseteq A$.
- The intersection of two sets, $A \cap B$ is the set of elements that are contained in both A and B , e.g.: $\{4, 1, x\} \cap \{1, 9, -2\} = \{1\}$.
- $\emptyset = \{\}$ is the empty set, containing no elements.
- Two sets, A, B , are disjoint if $A \cap B = \emptyset$.
- In probability and statistics, we often denote Ω as the sample space. This is the set of all possible outcomes.
- In probability and statistics, sets are also called events. In this case Ω is the certain event (which occurs with probability 1) and \emptyset is the null event (which occurs with probability 0).
- \bar{A} is the complement of the set A , it contains all elements $x \notin A$.
- $A \cup \bar{A} = \Omega$
- $A \cap \bar{A} = \emptyset$
- $\overline{A \cap B} = \bar{A} \cup \bar{B}$ (De Morgan's law)
- $\overline{A \cup B} = \bar{A} \cap \bar{B}$ (De Morgan's law)

5) Draw Venn diagrams for

- a. The union of two events.
- b. The intersection of two events.
- c. The intersection of three events.
- d. The complement of the intersection of two events (illustrate De Morgan's law).

6) Show that $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$.

7) Partitioning a set:

- a. For sets A and B_1, B_2, B_3, \dots such that $B_i \subseteq \Omega$ and $B_i \cap B_j = \emptyset$ for all i, j where $i \neq j$ and $\bigcup_i B_i = \Omega$, show that:

$$\bigcup_i (B_i \cap A) = A$$

- b. Show that $A = (B \cap A) \cup (\bar{B} \cap A)$.

Basic combinatorics:

- The multiplication principle: Example: Let $A = \{1,2,3,4,5,6\}$ be the set of outcomes of a die throw and $B = \{H, T\}$ the number of outcomes of a coin flip. Then the number of results in the experiment of throwing a die and flipping is, $|A||B| = 12$.
- The number of ways to order elements of a set A , with $|A| = n$ is $n!$ (factorial):

$$n! = \begin{cases} n \cdot (n-1) \cdot (n-2) \cdot \dots \cdot 2 \cdot 1 & \text{for } n \geq 1 \\ 1 & \text{for } n = 0 \end{cases}$$

- The number of ways to order a subset of size k of a set of size n , is,

$$n \cdot (n-1) \cdot (n-2) \cdot \dots \cdot (n-k+1) = \frac{n!}{(n-k)!}$$

- Denote by $\binom{n}{k}$ the number of subsets of size k of a set of size n , then,

$$\binom{n}{k} k! = \frac{n!}{(n-k)!},$$

thus,

$$\binom{n}{k} = \frac{n!}{(n-k)! k!}$$

- $\binom{n}{0} = 1, \binom{n}{n} = 1$.
- $\binom{n}{1} = n, \binom{n}{n-1} = n$.
- $\binom{n}{k} = \binom{n}{n-k}$.
- $(a+b)^n = \sum_{k=0}^n \binom{n}{k} a^k b^{n-k}$ (this is called the binomial formula).
- The total number of subsets of a set of size n , is 2^n . Prove it using the binomial formula. Prove it using the multiplication principle.

8) Write down the values of $\binom{7}{k}, k = 0,1,2,3,4,5,6,7$.

9) Which of the following two numbers is larger: $\binom{93}{30}$ or $\binom{93}{31}$?

10) Prove and explain (for $k \leq n$):

$$\binom{n}{k} + \binom{n}{k-1} = \binom{n+1}{k} \text{ (this is called Pascal's identity).}$$

Integration:

Let $f(x)$ be a non negative function defined over the reals. $\int_a^b f(x)dx$ is the area between the curve of $f(x)$ and the horizontal axis.

Example, $f(x) = \begin{cases} (d-c)^{-1} & c \leq x \leq d \\ 0 & \text{otherwise} \end{cases}$

in this case, $\int_{-\infty}^{\infty} f(x)dx = 1$, and $\int_{-\infty}^{(c+d)/2} f(x)dx = 1/2$, and $\int_c^d x f(x)dx = \frac{c+d}{2}$.

Sometimes $\int_a^b f(x)dx$ can not be evaluated explicitly but needs to be done numerically. One rough way to do this is,

$$\int_a^b f(x)dx \approx \sum_{i=1}^n \frac{b-a}{n} f(a + i \frac{b-a}{n})$$

As $n \rightarrow \infty$ the sum converges to the value of the integral.

- 11) Let $f(x) = \sqrt{1-x^2}$ for $0 \leq x \leq 1$. It is known that the derivative of $\frac{1}{2}(x\sqrt{1-x^2} + \text{ArcSin}(x))$ is $f(x)$. Find $\int_0^1 f(x)dx$. How does your result relate to the area of a circle? (reminder: The points, (x,y) on the unit circle satisfy $x^2 + y^2 = 1$.)
- 12) Let $f(x) = \begin{cases} 1+x & -1 \leq x \leq 0 \\ 1-x & 0 < x \leq 1 \\ 0 & \text{otherwise} \end{cases}$, find
- Find $\int_{-\infty}^{\infty} f(x)dx$ using simple geometry.
 - Check your answer by carrying out the integral explicitly (as studied in basic calculus).
 - Use Excel or similar software to numerically obtain the answer. Plot the accuracy of your result as a function of n (the number of elements in the approximating sum).
- 13) Let $f(x) = 2e^{-2x}$ for $0 < x$, and 0 when x is negative. Find the function $F(x) = \int_{-\infty}^x f(u)du$.
- 14) Let $f(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}$. Find numerically $\int_{-\infty}^0 f(x)dx$. Using your result, what is $\int_{-\infty}^{\infty} f(x)dx$?

Selected Solutions

2) $\sum_{i=1}^{100} (5i + 2) = 5 \sum_{i=1}^{100} i + 200 = 5 \frac{100(101)}{2} + 200 = 25450$

3) 37520

4) $2^n - 1$

5) 1,7,21,35,35,21,7,1

9) Both numbers have 93! in the numerator, so it is the denominator that makes the difference. The left number has 63!30! in the denominator and the right number has 62!31! in the denominator. The left denominator is bigger so the right number is bigger.

10) Proof 1: Make a common denominator to the left side and arrive to the right side.

Proof 2: The number on the right is the number of subsets of size k from a team of n players + captain.

The left is the sum of two things: $\binom{n}{k}$ is the number of subsets that do not include the captain. $\binom{n}{k-1}$ is the number of subsets that do include the captain. So the left and right sides are equal.

11) $\int_0^1 f(x)dx = \pi/4$ this is 1/4'th of the area of the unit circle.