Stochastic Modeling for Engineers
HW Project Number 1: 40 Probability Questions
Outline of Solutions

Yoni Nazarathy

September 12, 2011

Selected

2.9 $A=(-\infty, r]$ $r \le s$ $A^c = (r, \infty)$

$B=(-\infty, s]$ $B^c = (s, \infty)$

$C=(r,s]=B \setminus A = B \cap A^c$

$A \cap C = A \cap (B \cap A^c) = (A \cap A^c) \cap B = \emptyset \cap B = \emptyset$

$A \cup C = A \cup (B \cap A^c) = (A \cup B) \cap (A \cup A^c) = (A \cup B) \cap S = A \cup B = B$

$A \subseteq B$

2.14 a) $(A \cap B^c \cap C^c) \cup (A^c \cap B \cap C^c) \cup (A^c \cap B^c \cap C)$

b) $(A^c \cap B \cap C) \cup (A \cap B^c \cap C) \cup (A \cap B \cap C^c)$

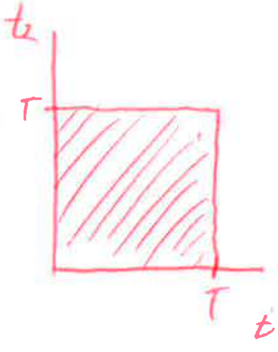
e) $A^c \cap B^c \cap C^c$

c) $A \cup B \cup C = (e)^c$

d) (b) $\cup$ (c).

(2.18)

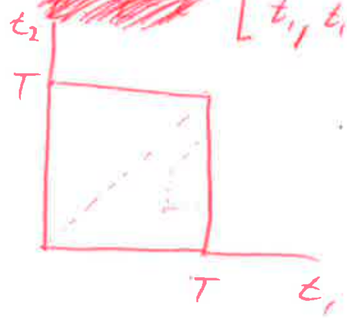
(a)

$S =$  $= \{(x,y) : 0 \le x, y \le T\}$

(b) collision $\Longleftrightarrow$ Car and train in same spot

$\Longleftrightarrow [t_1, t_1+d_1] \cap [t_2, t_2+d_2] \neq \emptyset$

$\Longleftrightarrow$



(2.23) $P(c,d) = \frac{3}{8}$, $P(d) = \frac{1}{8} \Longrightarrow P(c) = \frac{2}{8} = \frac{1}{4}$

$P(b,c) = \frac{6}{8}$, $P(c) = \frac{2}{8} \Longrightarrow P(b) = \frac{4}{8} = \frac{1}{2}$

$P(a,b,c,d) = 1 \Longrightarrow P(a) = 1 - \frac{4}{8} - \frac{2}{8} - \frac{1}{8} = \frac{1}{8}$

3.90

$X_i \sim exp(\lambda)$, $\frac{1}{\lambda} = 1$, $\lambda = 1$

$Y_n = X_1 + \dots + X_n \sim Gamma(n, 1)$

"Erlang"

$\underline{P}(Y_n > 2) = 0.9$

$$1 - \int_0^2 \frac{1}{(n-1)!} x^{n-1} e^{-x} dx = 0.9$$

$\Downarrow$

solve for n. (numerically)

(3.89)

$X \equiv$ # of messages sent untill success

$X \sim Geom(1-p)$ $\qquad E[X] = \frac{1}{1-p}$

$Y \equiv$ Duration untill success

$$Y = 2T \cdot X$$

$$E[Y] = 2T\,E[X] = \frac{2T}{1-p}$$

$$Rate = \frac{1}{E[Y]} = \frac{1-p}{2T}$$

4.172

Type I w.p $\frac{4}{8}$ $\quad exp(\lambda)$, $\frac{1}{\lambda}=2$, $\lambda=\frac{1}{2}$

Type II w.p $\frac{1}{8}$ $\quad$ Pareto$(\alpha=3, X_m=1)$

Type III w.p $\frac{3}{8}$ $\quad$ constant $=2$

$$P(X>15) = \sum_{types} P(X>15 \mid type) P(type)$$

$$= \frac{4}{8} P(exp(\tfrac{1}{2})>15) + \frac{1}{8} P(Pareto(\alpha=3, X_m=1)>15) + \frac{3}{8}\cdot 0$$

$$= \frac{4}{8} e^{-\frac{1}{2}\cdot 15} + \frac{1}{8}\left(\frac{1}{15}\right)^3 = 0.00031$$

see e.g. wikipedia Pareto Distribution

Markov Inequality:

$$P(X>15) \leq \frac{EX}{15} = \frac{\frac{4}{8}\cdot 2 + \frac{1}{8}\left(\frac{3\cdot 1}{3-1}\right) + \frac{3}{8}\cdot 2}{15} = 0.1242$$

4.173

$$P(X_1>9 \cup X_2>9 \cup X_3>9 \mid X_1>1, X_2>1, X_3>1) =$$

$$= 1 - P(X_1<9, X_2<9, X_3<9 \mid X_1>1, X_2>1, X_3>1)$$

$$= 1 - P(\cancel{X_1}<9 \mid X>1)^3 = 1 - \left(\frac{P(X<9, X>1)}{P(X>1)}\right)^3$$

$$= 1 - \frac{P(1<X<9)^3}{(2/2+1)^3}$$

$$= 1 - \frac{\left(\frac{2}{2+1} - \frac{2}{2+9}\right)^3}{(2/3)^3} = 0.385$$

4.174

$Y = \max\{X_1, \dots, X_n\}$

$$F_X(x) = \begin{cases} 0 & x \le 0 \\ \frac{1}{a}x & 0 < x < a \\ 1 & a \le x \end{cases}$$

(a) $P(Y \le y) = P(\max\{X_1, \dots, X_n\} \le y) = P(X_1 \le y, \dots, X_n \le y)$

$$= P(X_1 \le y) \cdot \dots \cdot P(X_n \le y) = P(X_1 \le y)^n = \begin{cases} 0 & y \le 0 \\ \left(\frac{1}{a}\right)^n y^n & 0 \le y \le a \\ 1 & a \le y \end{cases}$$

$f_Y(y) = \frac{n}{a^n} y^{n-1} \quad y \in [a, 1]$

(b) $E[Y] \int_0^a y \frac{n}{a^n} y^{n-1} dy$

$= \frac{an}{n+1} = \frac{n}{n+1} a$

$E[Y^2] = \int_0^a y^2 \frac{n}{a^n} y^{n+1}$

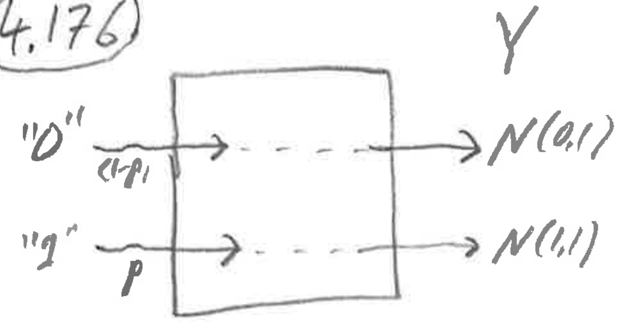
$= \frac{a^2 n}{n+2}$

$Var(Y) = E[Y^2] - E^2[Y] = \frac{n}{(n+2)(n+1)^2} a^2$

As $n \to \infty$ $E[Y] \to a$ and $Var(Y) \to 0$

that is good.

4.176



(a) $$P(\text{"1"} \mid y < Y < y+h) = \frac{P(N(1,1) \in [y, y+h])\, p}{(1-p) P(N(0,1) \in [y,y+h]) + p P(N(1,1) \in [y, y+h])}$$

Bayes

$$= \frac{p\left(\phi(y+h-1) - \phi(y-1)\right)}{(1-p)\left(\phi(y+h) - \phi(y)\right) + p\left(\phi(y+h-1) - \phi(y-1)\right)}$$

for h small

$$\cong \frac{p\, e^{-\frac{(y-1)^2}{2}}}{(1-p)\, e^{-\frac{y^2}{2}} + p\, e^{-\frac{(y-1)^2}{2}}}$$

Similarly

$$P(\text{"0"} \mid y < Y < y+h) \cong \frac{(1-p) e^{-\frac{y^2}{2}}}{(1-p) e^{-\frac{y^2}{2}} + p\, e^{-\frac{(y-1)^2}{2}}}$$

(b)

$$p\, e^{-\frac{(y-1)^2}{2}} \;\substack{\le \\ = \\ >}\; (1-p) e^{-\frac{y^2}{2}}$$

$$\log p - \frac{(y-1)^2}{2} \;\substack{\le \\ = \\ >}\; \log(1-p) - \frac{y^2}{2}$$

$$\log \frac{p}{1-p} \;\substack{\le \\ = \\ >}\; -\frac{1}{2}\left(y^2 - (y-1)^2\right) = -\frac{1}{2}\left(y^2 - y^2 + 2y - 1\right) = -y + \frac{1}{2}$$

$$T = \frac{1}{2} - \log \frac{p}{1-p} \;\substack{\ge \\ = \\ <}\; y$$

(c) Use normal curve.