Stochastic Modeling for Engineers
Test: Probability

Lecturer: Yoni Nazarathy

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Yes/No Questions

Answer each question with "yes" or "no" and justify your reason very briefly.

- Let X be a random variable with known mean m and known standard deviation σ, then it is possible to determine the CDF of X uniquely.
  Yes / No. Reason: The mean and variance do not uniquely identify the CDF.

- If X is a random variable and $g(x)$ is a real function, then necessarily $E[g(X)] = g(E[X])$.
  Yes / No. Reason: This is true if $g(x) = ax+b$, but not in general.

- If X is a random variable and $Var(X) = 0$ then there exists a constant c, such that, $P(X = c) = 1$.
  Yes / No. Reason: X is a deterministic (degenerate) RV.

- The mean of a discrete random variable taking values $\{0, 1, 2, 3, 4\}$ is necessarily an integer.
  Yes / No. Reason: $EX = \sum x p_X(x)$ can be non-integer.

- Take a CDF $F(x)$ that is one-to-one and has an inverse $F^{-1}(x)$. Let U be a uniform $[0, 1]$ random variable. Then the CDF of $F^{-1}(|2U - 1|)$ is $F(x)$. (Here $|z|$ denotes the absolute value of z).
  Yes / No. Reason:

  $2U - 1 \sim \text{Uniform}(-1,1)$

  $|\text{Uniform}(-1,1)| \sim \text{Uniform}(0,1)$

  $F^{-1}(\text{Uniform}(0,1)) \sim F$

Open Questions

1. Let $X \sim \exp(\lambda)$. Define $Y = \lfloor X \rfloor$ ($\lfloor z \rfloor$ is the greatest integer that is less than or equal to z).
   (a) Find the distribution of Y in terms of either CDF or PMF.
   (b) Find $E[Y]$.

2. Assume X is a continuous random variable with CDF $F_X(x)$. Let $Y = F_X(X)$, find the distribution of Y in terms of either CDF or PDF.

3. The number X of accidents counted in an hour in Melbourne is a Poisson random variable with rate λ_1 when there is rain and a Poisson random variable with rate λ_0 when there is no rain. Assume it is raining with probability p.
   (a) Find the mean number of accidents in an hour.
   (b) Find $P[\text{it is raining}|X = k]$.

Good Luck.

(2) $F_Y(y) = P(Y \le y) = P(F_X(X) \le y)$

$0 \le y \le 1$ $= P(X \le F_X^{-1}(y))$

$= F_X(F_X^{-1}(y)) = y$

So $Y \sim \text{Uniform}(0,1)$

(3) (a) $p\lambda_1 + (1-p)\lambda_0$

(b) $P(\text{rain} | X=k) = \dfrac{\overbrace{P(X=k|\text{rain})}^{e^{-\lambda_1}\frac{\lambda_1^k}{k!}} \cdot \overbrace{P(\text{rain})}^{p}}{p e^{-\lambda_1}\frac{\lambda_1^k}{k!} + (1-p)e^{-\lambda_0}\frac{\lambda_0^k}{k!}}$

$= \dfrac{p e^{-\lambda_1}\lambda_1^k}{p e^{-\lambda_1}\lambda_1^k + (1-p)e^{-\lambda_0}\lambda_0^k}$.

(1a) $k = 0,1,2,\ldots$ $P(Y=k) = P(\lfloor X \rfloor = k)$

$= \int_k^{k+1} \lambda e^{-\lambda x} dx = (1-e^{-\lambda})(e^{-\lambda})^k$

$Y \sim \text{Geom}(1-e^{-\lambda})$
$0,1,2,\ldots$

(1b) So $EY = \dfrac{e^{-\lambda}}{1-e^{-\lambda}} = \dfrac{1}{e^{\lambda}-1}$

alternatively:

$EY = \int_0^{\infty} \lfloor x \rfloor \lambda e^{-\lambda x} dx$

$= \sum_{k=0}^{\infty} \int_k^{k+1} k \lambda e^{-\lambda x} dx$

$=$ same result