Lesson #8: Statistical Inference

In this lesson, we will almost exclusivly discuss estimation of a parameter in a parametric model.

Let X_1, X_2, ...,
X_n be a "random i.i.d. sample" from distribution $f(x_i \mid \theta)$.

We are interested in using the observations X_1,
X_2, ..., X_n to "guess" what is $g(\theta)$, often just θ.

Example :

Let $X_i \sim \text{Gamma}(\alpha, \lambda)$.

Here $\theta = \{\alpha, \lambda\}$. We may be interested in estimating $g(\theta) = \frac{\alpha}{\lambda}$,
the mean. Or, $g(\theta) = \{\theta\}$. Or $g(\theta) = \frac{\alpha}{\lambda^2}$, the variance, etc ..

An estimator, $\bar{\theta}$ (I wanted hat) is a function of X_1,
X_2, ..., X_n. It is thus a random variable.

Attributes of Estimators

- **Bias**

```
myVariance[xx_] := (Table[(xx[[i]] - Mean[xx])^2, {i, 1, Length[xx]}] // Total) / Length[xx]

nnn = 10 000;

rvs = RandomReal[NormalDistribution[], nnn];

myVariance[rvs]

0.981065

Table[myVariance[RandomReal[NormalDistribution[], 5]], {10 000}] // Mean

0.801724

(5 - 1)/5 // N

0.8
```

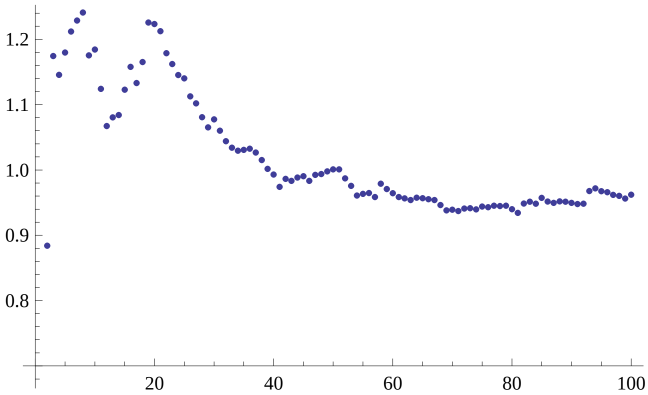
So this is why we use $\frac{1}{n-1}$ In denominator of sample variance

▪ Consistency

```
xx = RandomReal[NormalDistribution[], 10 000];

Table[myVariance[Take[xx, n]], {n, 10, 1000, 10}];

ListPlot[%]
```



Maximum Likelihood Estimation

▪ Example - Bernoullli(p)

```
PDF[BernoulliDistribution[p], x]
```

$$\begin{cases} 1-p & x == 0 \\ p & x == 1 \end{cases}$$

```
like[p_, xx_] := Apply[Times, Table[p^xx[[i]] (1 - p)^(1-xx[[i]]), {i, 1, Length[xx]}]]

like[p, {1, 0, 1}]
```

$(1 - p)\, p^2$

y is total of xx. n is the length

```
logLike[p_, y_] := y Log[p] + (n - y) Log[1 - p]

Clear[xx]

Solve[D[logLike[p, y], p] == 0, p]
```

$\left\{\left\{p \to \frac{y}{n}\right\}\right\}$

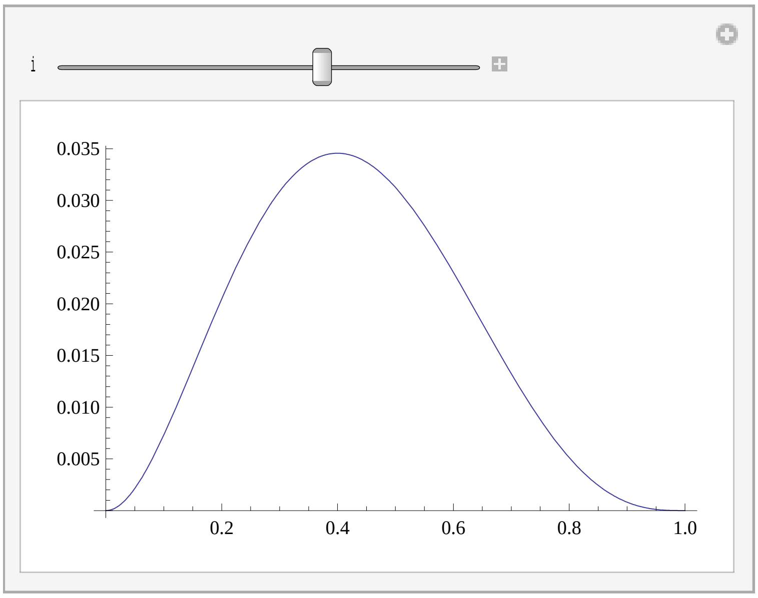
```
res = Tuples[{0, 1}, 5]

{{0, 0, 0, 0, 0}, {0, 0, 0, 0, 1}, {0, 0, 0, 1, 0}, {0, 0, 0, 1, 1},
 {0, 0, 1, 0, 0}, {0, 0, 1, 0, 1}, {0, 0, 1, 1, 0}, {0, 0, 1, 1, 1},
 {0, 1, 0, 0, 0}, {0, 1, 0, 0, 1}, {0, 1, 0, 1, 0}, {0, 1, 0, 1, 1},
 {0, 1, 1, 0, 0}, {0, 1, 1, 0, 1}, {0, 1, 1, 1, 0}, {0, 1, 1, 1, 1}, {1, 0, 0, 0, 0},
 {1, 0, 0, 0, 1}, {1, 0, 0, 1, 0}, {1, 0, 0, 1, 1}, {1, 0, 1, 0, 0}, {1, 0, 1, 0, 1},
 {1, 0, 1, 1, 0}, {1, 0, 1, 1, 1}, {1, 1, 0, 0, 0}, {1, 1, 0, 0, 1}, {1, 1, 0, 1, 0},
 {1, 1, 0, 1, 1}, {1, 1, 1, 0, 0}, {1, 1, 1, 0, 1}, {1, 1, 1, 1, 0}, {1, 1, 1, 1, 1}}
```

```
posLike = Map[like[p, #] &, res]
```

$\{(1-p)^5,\ (1-p)^4 p,\ (1-p)^4 p,\ (1-p)^3 p^2,\ (1-p)^4 p,\ (1-p)^3 p^2,\ (1-p)^3 p^2,\ (1-p)^2 p^3,$
$(1-p)^4 p,\ (1-p)^3 p^2,\ (1-p)^3 p^2,\ (1-p)^2 p^3,\ (1-p)^3 p^2,\ (1-p)^2 p^3,\ (1-p)^2 p^3,\ (1-p)\ p^4,$
$(1-p)^4 p,\ (1-p)^3 p^2,\ (1-p)^3 p^2,\ (1-p)^2 p^3,\ (1-p)^3 p^2,\ (1-p)^2 p^3,\ (1-p)^2 p^3,\ (1-p)\ p^4,$
$(1-p)^3 p^2,\ (1-p)^2 p^3,\ (1-p)^2 p^3,\ (1-p)\ p^4,\ (1-p)^2 p^3,\ (1-p)\ p^4,\ (1-p)\ p^4,\ p^5\}$

```
Manipulate[
 Plot[posLike[[i]], {p, 0, 1}],
 {i, 1, Length[res], 1}, SaveDefinitions → True]
```



Continue to make a demonstration ....

■ Example - Gamma(α,β)

```
PDF[GammaDistribution[α, β], x]
```

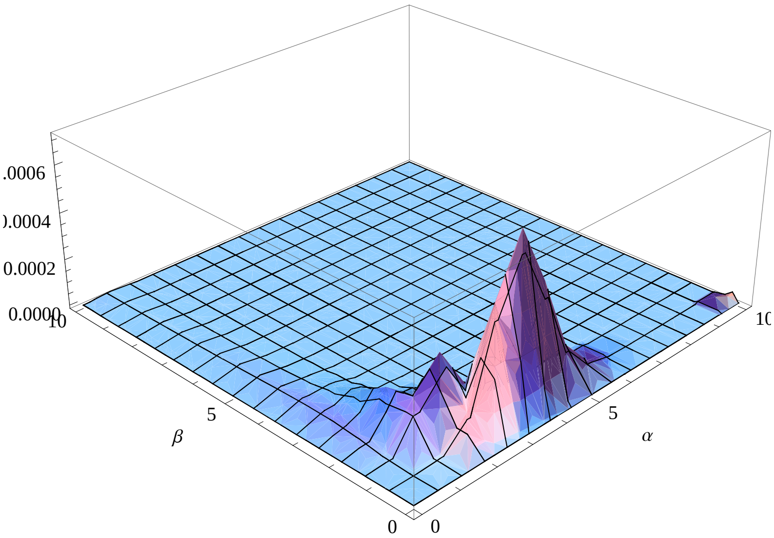
$$\frac{e^{-\frac{x}{\beta}}\, x^{-1+\alpha}\, \beta^{-\alpha}}{\text{Gamma}[\alpha]}$$

```
Mean[GammaDistribution[α, β]]
```

$\alpha\ \beta$

In[18]:= `like[α_, β_, xx_] := Apply[Times, Table[` $\frac{e^{-\frac{xx[[i]]}{\beta}}\, xx[[i]]^{-1+\alpha}\, \beta^{-\alpha}}{\text{Gamma}[\alpha]}$ `, {i, 1, Length[xx]}]]`

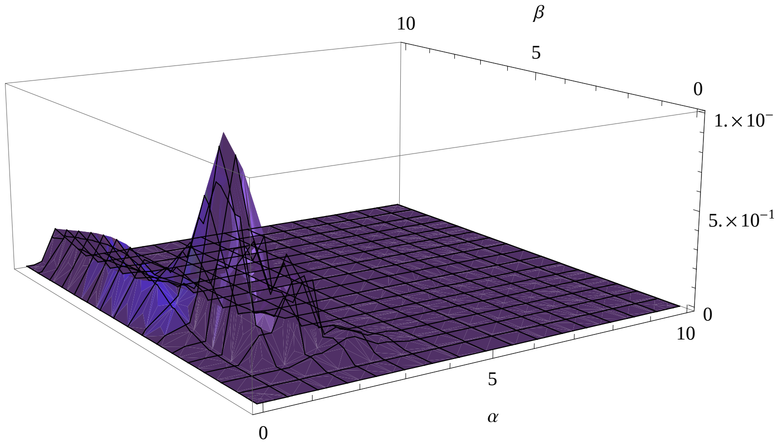
```
Plot3D[like[α, β, {2, 4, 1, 2, 3}], {α, 0, 10},
 {β, 0, 10}, PlotRange → All, AxesLabel → {α, β}]
```

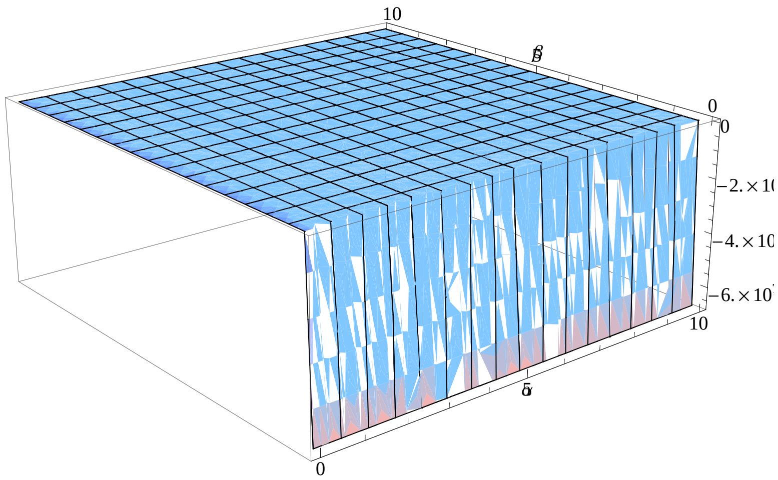



```
plotSample[xx_] :=
 Plot3D[like[α, β, xx], {α, 0, 10}, {β, 0, 10}, PlotRange → All, AxesLabel → {α, β}]

plotLogSample[xx_] :=
 Plot3D[Log[like[α, β, xx]], {α, 0, 10}, {β, 0, 10}, PlotRange → All, AxesLabel → {α, β}]
```

```
sample = RandomReal[GammaDistribution[2, 2], 10];
plotSample[sample]
plotLogSample[sample]
```





```
NMaximize[like[α, β, sample], {α, β}]
```

$\{1.02194\times10^{-11}, \{\alpha \to 1.41666, \beta \to 3.37364\}\}$

```
NMaximize[Log[like[α, β, sample]], {α, β}]
```

$\{-25.3067, \{\alpha \to 1.41666, \beta \to 3.37364\}\}$

Alternative : method of moments:

```
Mean[GammaDistribution[α, β]]
```

$\alpha\,\beta$

```
Variance[GammaDistribution[α, β]]
```

$\alpha\,\beta^2$

```
Solve[
 {Mean[GammaDistribution[α, β]] == xBar,
  Variance[GammaDistribution[α, β]] == sSquared}, {α, β}]
```

$$\left\{\left\{\alpha \to \frac{\text{xBar}^2}{\text{sSquared}}, \beta \to \frac{\text{sSquared}}{\text{xBar}}\right\}\right\}$$

```
sample
```

```
{2.61934, 9.20542, 2.46891, 0.863936, 2.68407, 0.7464, 3.60698, 16.146, 6.11415, 3.3378}
```

In[59]:=

$$\text{estMoments[sample\_]} := \left\{\frac{\text{Mean[sample]}^2}{\text{Variance[sample]}}, \frac{\text{Variance[sample]}}{\text{Mean[sample]}}\right\}$$

```
estMoments[sample]
```

```
{1.0278, 4.65001}
```

```
estMoments[RandomReal[GammaDistribution[2, 2], 100 000]]
```

```
{1.98464, 2.02088}
```

Now lets make a "serious" demonstration of this ...

In[102]:=

```
testAlpha = 1; testBetta = 1;
repeats = 30;
sampleSize = 2000;
samples =
  Table[RandomReal[GammaDistribution[testAlpha, testBetta], sampleSize], {repeats}];
logLikes = Map[Log[like[α, β, #]] &, samples];
estsMLE = Map[({α, β} /. NMaximize[#, {α, β}][[2]]) &, logLikes];
estsMoment = Map[estMoments, samples];
```

In[109]:=

```
ListPlot[{estsMLE, estsMoment}, AxesOrigin → {0.8, 0.8}]
```

Out[109]=