



A correction term for the covariance of renewal-reward processes with multivariate rewards



Brendan Patch*, Yoni Nazarathy, Thomas Taimre

The School of Mathematics and Physics, The University of Queensland, Qld 4072, Australia

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ABSTRACT

We consider a renewal-reward process with multivariate rewards. Such a process is constructed from an i.i.d. sequence of time periods, to each of which there is associated a multivariate reward vector. The rewards in each time period may depend on each other and on the period length, but not on the other time periods. Rewards are accumulated to form a vector valued process that exhibits jumps in all coordinates simultaneously, only at renewal epochs.

We derive an asymptotically exact expression for the covariance function (over time) of the rewards, which is used to refine a central limit theorem for the vector of rewards. As illustrated by a numerical example, this refinement can yield improved accuracy, especially for moderate time-horizons.

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1. Introduction

Probabilistic modelling and analysis has a long tradition in dealing with the behaviour of regenerative processes. Such processes restart probabilistically at renewal instances, forming a sequence of independent and identically distributed (i.i.d.) sub-processes. They are used in stochastic simulation, reliability analysis, actuarial studies, queueing theory, and other aspects of applied probability and statistics. An illustrative example is that of natural disasters. When a natural disaster occurs there are several simultaneous costs (e.g. personal property loss and infrastructure damage), which may be distributed across different locations. These costs (rewards) are typically dependent and may also depend on the time elapsed since the previous disaster. If we assume that the system resets after such an event then the situation is well described by a renewal-reward process with multivariate rewards – which we will refer to simply as a *multivariate renewal-reward process*.

The multivariate renewal-reward process is constructed on a probability space supporting $\{\mathbf{Z}_n\}_{n=0}^{\infty}$, a sequence of $(L+1)$ -dimensional independent random vectors with possibly dependent coordinates. The first coordinate of \mathbf{Z}_n , denoted T_n , signifies the time between events, which we call *renewals*, and is assumed non-negative. The remaining L coordinates, denoted $X_{1,n}, \dots, X_{L,n}$ are the *rewards* and are not sign restricted. Assume that $\{\mathbf{Z}_n\}_{n=1}^{\infty}$ are i.i.d. and, as is standard in renewal theory (see e.g. [Asmussen, 2003](#) or [Gut, 2009](#)), \mathbf{Z}_0 may follow a different distribution. We refer to the case of $T_0 \equiv 0$ and all $X_{i,0} \equiv 0$ as *ordinary*; otherwise the process is *delayed*. To avoid trivialities assume that T_n and all $X_{i,n}$ are almost surely not zero for $n \geq 1$.

* Corresponding author.

E-mail address: b.patch@uq.edu.au (B. Patch).

Let $S_n \stackrel{\text{def}}{=} \sum_{i=0}^n T_i$, so that $\{S_n\}_{n=0}^{\infty}$ are the renewal times. Taking $N(t) \stackrel{\text{def}}{=} \min\{n : S_n > t\}$ the multivariate renewal-reward process is $\{\mathbf{R}(t) : t \geq 0\}$, or simply $\mathbf{R}(\cdot)$, where

$$\mathbf{R}(t) \stackrel{\text{def}}{=} \left[\sum_{n=0}^{N(t)-1} X_{1,n}, \dots, \sum_{n=0}^{N(t)-1} X_{L,n} \right]. \quad (1)$$

We treat the summations in $\mathbf{R}(t)$ as empty for $t < T_0$, since $N(t) = 0$ there. For $L > 2$ the i th coordinate of $\mathbf{R}(\cdot)$ is denoted $R_i(\cdot)$. For $L = 2$ the coordinates are $R_x(\cdot)$ and $R_y(\cdot)$, and the n -th reward vector $[X_{1,n}, X_{2,n}]$ is written simply as $[X_n, Y_n]$, where we represent all vectors as rows. Note that in the ordinary case, $N(0) = 1$ and in the delayed case $N(0) = 0$.

We focus on the case of moderate or large t and aim to approximate the distribution of $\mathbf{R}(t)$. In the illustrative example of natural disasters, this is the multivariate distribution describing the different types of losses accumulated during the first $[0, t]$ time units. In some very special cases the distribution of $\mathbf{R}(t)$ admits an explicit form. If, for example, the coordinates of \mathbf{Z}_n are mutually independent and T_n is exponentially distributed, then $N(\cdot)$ is a Poisson process and $\mathbf{R}(\cdot)$ is a vector of independent compound Poisson processes. In general, however, the distribution of $\mathbf{R}(t)$ is not easily obtainable, in which case asymptotic approximations become particularly appealing.

Under regularity conditions (described in the next section) it is well known that $\mathbf{R}(t)$ obeys a normal central limit theorem (CLT) as $t \rightarrow \infty$, where the mean and covariance terms appearing in the CLT are determined by moments of \mathbf{Z}_1 . Brown and Solomon further established in Brown and Solomon (1975) that for a renewal-reward process satisfying suitable regularity conditions with univariate rewards ($L = 1$),

$$\mathbb{E} R_x(t) = a_x t + b_x + o(1) \quad \text{and} \quad \text{Var}(R_x(t)) = c_x t + d_x + o(1),$$

where $o(1)$ is a function that vanishes as $t \rightarrow \infty$. Here, the constants a_x and c_x are determined by moments (including cross moments) of $[T_1, X_1]$ and the constants b_x and d_x are determined by moments of $[T_0, X_0]$ and $[T_1, X_1]$. Expressions for a_x and c_x , as well as a version of b_x with rewards independent of renewals, were found by Smith in Smith (1955). Subsequently, in Brown and Solomon (1975), Brown and Solomon extended to find b_x and d_x for the general univariate renewal-reward process.

The main contribution of the current paper is to generalize the result of Brown and Solomon (1975) to multivariate rewards. We prove that, under regularity conditions,

$$\text{Cov}(R_x(t), R_y(t)) = c_{x,y} t + d_{x,y} + o(1).$$

As before, $c_{x,y}$ depends on the moments of \mathbf{Z}_1 and $d_{x,y}$ depends on the moments of both \mathbf{Z}_1 and \mathbf{Z}_0 . Expressions for $c_{x,y}$ appeared in Smith (1955), although without an explicit proof for this form of the covariance curve. Our expression for $d_{x,y}$ is new and generalizes d_x of Brown and Solomon (1975).

The multivariate CLT for $\mathbf{R}(t)$ first appeared in Smith (1955). The CLT uses a covariance matrix with elements $c_{x,y}$ (or c_x on the diagonal). Our refined asymptotics suggest an improved approximation to $\mathbf{R}(t)$ based on this CLT, our new $d_{x,y}$ term, and the previously known d_x term. We illustrate the usefulness of this improved approximation in an example. A further (minor contribution) of the current paper is in casting Smith's CLT in a modern form. A related presentation is in Section 7.4 of Whitt (2002), where functional CLTs are given. The case handled there is one dimensional and assumes that rewards are independent of renewals.

The remaining part of the paper is structured as follows. In Section 2 we present our main result on correction terms to the covariance curve of the multivariate renewal-reward process as well as the CLT and the improved approximation to $\mathbf{R}(t)$. Section 3 demonstrates the usefulness of our correction terms through a numerical illustration. Proofs are in Section 4. We conclude in Section 5.

2. Main results

Our results are stated in terms of moments (and cross moments) of \mathbf{Z}_0 and \mathbf{Z}_1 . It is useful to denote some of the moments of \mathbf{Z}_1 as follows: $\mu_i \stackrel{\text{def}}{=} \mathbb{E} T_1^i$, $\lambda_i \stackrel{\text{def}}{=} \mathbb{E} X_1^i$, $\alpha_i \stackrel{\text{def}}{=} \mathbb{E} Y_1^i$, $m_{i,j} \stackrel{\text{def}}{=} \mathbb{E} T_1^i X_1^j$, $n_{i,j} \stackrel{\text{def}}{=} \mathbb{E} T_1^i Y_1^j$, and $p_{i,j,k} \stackrel{\text{def}}{=} \mathbb{E} T_1^i X_1^j Y_1^k$. Denote the distribution function of T_0 by $F_0(\cdot)$ and that of T_1 (and subsequent inter-event times) by $F(\cdot)$. We call $F(\cdot)$ non-lattice if the corresponding probability measure $dF(\cdot)$ is not concentrated on a set of the form $\{\delta, 2\delta, \dots\}$. A distribution function is said to have the stronger property of being spread out if $F^{(n)}(\cdot)$, the n -th convolution of $F(\cdot)$, has a component that is absolutely continuous (e.g. Asmussen, 2003, Section 7.1).

The growth rate $a_x \stackrel{\text{def}}{=} \lambda_1 / \mu_1$ is well known. In Brown and Solomon (1975), it was further established:

Theorem 1 (Restatement of Brown and Solomon, 1975, Lemma 1). For $F(\cdot)$ non-lattice and μ_2, λ_1 , and $m_{1,1}$ finite,

$$\mathbb{E} R_x(t) = a_x t + b_x + o(1), \quad (2)$$

where $a_x \stackrel{\text{def}}{=} \lambda_1 / \mu_1$ and $b_x \stackrel{\text{def}}{=} \mu_1^{-1} \mu_2 a_x / 2 - \mu_1^{-1} m_{1,1} + \mathbb{E} X_0 - a_x \mathbb{E} T_0$.

In order to state our main result, consider an ordinary renewal-reward process where the rewards are distributed as the product $X_1 Y_1$. For ordinary $\mathbf{R}(\cdot)$ denote the i th reward coordinate, $\sum_{n=1}^{N(t)-1} X_{i,n}$, by $\dot{R}_i(\cdot)$. In particular, for $L = 2$, we write $\dot{R}_x(\cdot)$, $\dot{R}_y(\cdot)$, and $\dot{R}_{xy}(\cdot)$ for the two reward coordinates and the associated product reward coordinate. Applying [Theorem 1](#) above, we have that $\mathbb{E} \dot{R}_{xy}(t) \stackrel{\text{def}}{=} \mathbb{E} \sum_{n=1}^{N(t)-1} X_n Y_n$ can be represented as

$$\mathbb{E} \dot{R}_{xy}(t) = a_{xy} t + \dot{b}_{xy} + o(1), \quad \text{with } a_{xy} = \mu_1^{-1} p_{0,1,1} \quad \text{and} \quad \dot{b}_{xy} = \mu_1^{-1} a_{xy} \mu_2 / 2 - \mu_1^{-1} p_{1,1,1}. \tag{3}$$

Our main theorem is a generalization of the key results in [Brown and Solomon \(1975\)](#). It utilizes the expressions for a_{xy} , \dot{b}_{xy} , a_x , as well as $a_y = \alpha_1 / \mu_1$, and the corresponding correction terms in the ordinary case,

$$\dot{b}_x = \mu_1^{-1} a_x \mu_2 / 2 - \mu_1^{-1} m_{1,1} \quad \text{and} \quad \dot{b}_y = \mu_1^{-1} a_y \mu_2 / 2 - \mu_1^{-1} n_{1,1}.$$

Theorem 2. For $F(\cdot)$ spread out and $\mu_3, \lambda_2, \alpha_2, m_{1,2}, n_{1,2}, p_{1,1,1}, \mathbb{E} T_0^2$, and $\mathbb{E} X_0 Y_0$ finite,

$$\text{Cov}(R_x(t), R_y(t)) = c_{x,y} t + d_{x,y} + o(1),$$

where

$$c_{x,y} \stackrel{\text{def}}{=} \mu_1^{-1} \text{Cov}(X_1 - a_x T_1, Y_1 - a_y T_1) = a_{xy} + a_x \dot{b}_y + a_y \dot{b}_x.$$

Further,

$$d_{x,y} \stackrel{\text{def}}{=} \dot{d}_{x,y} - c_{x,y} \mathbb{E} T_0 + a_x a_y \text{Var}(T_0) + \text{Cov}(X_0, Y_0) - a_x \text{Cov}(T_0, Y_0) - a_y \text{Cov}(T_0, X_0), \tag{4}$$

with

$$\dot{d}_{x,y} \stackrel{\text{def}}{=} \dot{b}_x \dot{b}_y + \dot{b}_{xy} + 2 a_x \ell_y + 2 a_y \ell_x,$$

where,

$$\begin{aligned} \ell_x &\stackrel{\text{def}}{=} \mu_1^{-3} \lambda_1 \mu_2^2 / 4 - \mu_1^{-2} \lambda_1 \mu_3 / 6 + \mu_1^{-1} m_{2,1} / 2 - \mu_1^{-2} \mu_2 m_{1,1} / 2, \\ \ell_y &\stackrel{\text{def}}{=} \mu_1^{-3} \alpha_1 \mu_2^2 / 4 - \mu_1^{-2} \alpha_1 \mu_3 / 6 + \mu_1^{-1} n_{2,1} / 2 - \mu_1^{-2} \mu_2 n_{1,1} / 2. \end{aligned} \tag{5}$$

Note that: (i) as shown in [Lemma 2](#) below, the quantity ℓ_x (as well its y -counterpart) is in fact the integrated $o(1)$ term of [Eq. \(2\)](#); (ii) for $y = x$ [Theorem 2](#) reduces to results of [Brown and Solomon \(1975\)](#) with $\dot{b}_{xy} = \dot{b}_{xx} = \mu_1^{-2} \mu_2 \lambda_2 / 2 - \mu_1^{-1} m_{1,2}$; and (iii) for ordinary $\mathbf{R}(\cdot)$ the terms involving \mathbf{Z}_0 vanish, implying $d_{x,y} = \dot{d}_{x,y}$.

For L -dimensional $\mathbf{R}(\cdot)$, we define the matrices and vectors:

$$\mathbf{a} = [a_i]_{i=1}^L, \quad \mathbf{b} = [b_i]_{i=1}^L, \quad C = \mu_1^{-1} \text{Cov}([\gamma_i]_{i=1}^L), \quad \text{and} \quad D = [d_{ij}]_{i,j=1}^L.$$

Here the elements a_i , b_i , and d_{ij} are as defined in [Theorems 1](#) and [2](#) above, where x and/or y are replaced by some pair $i, j \in \{1, \dots, L\}$, and $\gamma_i = X_{1,i} - a_i T_1$ for $i = 1, \dots, L$. The vector \mathbf{a} and the covariance matrix C play a role in the CLT which we state now. The vector \mathbf{b} and our (new contribution) matrix D are the *correction terms*. These appear in the refinement that follows.

Theorem 3 (Originally in [Smith, 1955](#)). If $F(\cdot)$ is spread out, $\mathbb{E} X_{i,1} T_1 < \infty$, and $\mathbb{E} X_{i,1} X_{j,1} < \infty$ then the sequence (in t) of random vectors,

$$\left[\frac{R_1(t) - a_1 t}{\sqrt{t}}, \dots, \frac{R_L(t) - a_L t}{\sqrt{t}} \right], \quad t > 0,$$

converges in distribution, as $t \rightarrow \infty$, to a zero mean normal random vector with covariance matrix C , denoted here by $\mathbf{N}(\mathbf{0}, C)$.

Motivated by [Theorems 1–3](#), we suggest the following *refined normal approximation* to the distribution of the multivariate renewal-reward process at time t :

$$\mathbf{R}(t) \stackrel{d}{\approx} \mathbf{N}(\mathbf{a} t + \mathbf{b}, C t + D). \tag{6}$$

Note that the matrix D may not be positive definite (PD) – in other words not a covariance matrix – whereas C always is. When D is not PD, it is easy to see that $C t + D$ is PD for all t greater than some $t_0 > 0$, and is not PD for all $t \leq t_0$. Consequently, we only suggest [\(6\)](#) when $C t + D$ is PD (i.e. $t > t_0$).

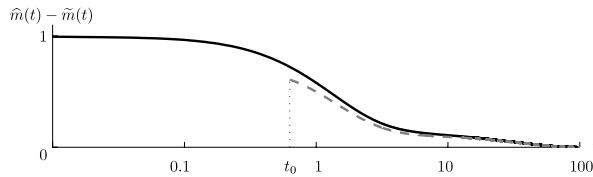


Fig. 1. Difference between two analytical approximations $\tilde{m}(t)$ and simulation estimate $\hat{m}(t)$. The dashed curve uses D , while the solid curve does not. The approximations $\tilde{m}(t)$ are easy to evaluate using (7) together with our proposed approximation (6).

3. Numerical illustration

To illustrate the applicability of our refined normal approximation (6), assume that we wish to evaluate

$$m(t) \stackrel{\text{def}}{=} \mathbb{E} \min\{R_x(t), R_y(t)\}.$$

A simple expression for this expected minimum is generally not available, but by approximating the distribution of $[R_x(t), R_y(t)]$ as normal using Theorem 3 we obtain a very good approximation for $m(t)$, which is generally improved using our refinement (6). The expected minimum of a bivariate normal random vector $[W, V]$ is

$$\Phi\left(\frac{\mathbb{E}(V - W)}{\text{Var}(W - V)}\right) \mathbb{E}W + \phi\left(\frac{\mathbb{E}(W - V)}{\text{Var}(W - V)}\right) \mathbb{E}V - \phi\left(\frac{\mathbb{E}(W - V)}{\text{Var}(W - V)}\right) \text{Var}(W - V), \tag{7}$$

where Φ and ϕ are respectively the cdf and pdf of the standard normal distribution (see e.g. Hunter, 1974). Thus, using the mean and variance/covariance expansions from Theorems 1 and 2 we can (for fixed t) combine (7) with (6) to get an explicit approximation of $m(t)$, denoted $\tilde{m}(t)$. For moderate to large t , we expect using $Ct + D$ as the covariance will yield a better approximation of $m(t)$ than only using Ct . That is, we expect that the matrix D , with our newly found covariance refinement term $d_{x,y}$ on the off-diagonals, will improve the approximation.

As a specific numerical example consider

$$\mathbf{Z}_n = [T_n, X_n, Y_n] = [U_{1,n} + U_{4,n}, U_{2,n} + U_{4,n}, U_{3,n} + U_{4,n}], \quad n = 0, 1, 2, \dots,$$

where $U_{i,n}$ are all independent exponential random variables with unit mean for $i = 1, 2, 4$ and mean $1/2$ for $i = 3$. Now, using the expressions of Theorems 1 and 2 we obtain

$$\mathbf{a} = [1, -1], \quad \mathbf{b} = [-1, -8/7], \quad C = \begin{bmatrix} 1, & 3/8 \\ 3/8, & 7/16 \end{bmatrix}, \quad \text{and} \quad D = \begin{bmatrix} 1/2, & 1/2 \\ 1/2, & 13/64 \end{bmatrix}.$$

In this case the refinement to the covariance curve is only applicable for $t > t_0 = (\sqrt{731} - 3)/38 \approx 0.63$, since $Ct + D$ is not a PD matrix when $t \leq t_0$. We estimated the true $m(t)$ by extensive simulation, taking the mean of the minimum over 10^7 sample paths of $\mathbf{R}(\cdot)$ as the estimate $\hat{m}(t)$.

Fig. 1 plots the difference between the estimated (true value) $\hat{m}(t)$ and two versions of the approximate $\tilde{m}(\cdot)$. The solid curve does not use the correction matrix D , i.e. $\tilde{m}(t)$ is evaluated assuming covariance Ct . The dashed curve is the improved approximation incorporating an asymptotically exact covariance, $Ct + D$ (both curves utilize \mathbf{b}). As observed, while both curves converge to zero error as $t \rightarrow \infty$, the refinement yields smaller error – especially in the “medium” time horizon after t_0 .

4. Proof of main result

Without loss of generality we prove Theorem 2 as stated for the case $L = 2$. In the ordinary case denote $M(t) \stackrel{\text{def}}{=} \mathbb{E}N(t)$ and the closely related $D_x(t) \stackrel{\text{def}}{=} \mathbb{E}\dot{R}_x(t)$ (as well as the y -counterpart $D_y(t)$).

Lemma 1. For ordinary $\mathbf{R}(\cdot)$: $\mathbb{E} \sum_{i < j \leq N(t)-1} X_i Y_j = \int_0^t D_x(t - s) dD_y(s)$.

Proof. Let $f_n = dF^{(n)} / dM$. Since $M(t) = \sum_{n=0}^\infty F^{(n)}(t)$ for all t , $M(t) = 0$ implies all $F^{(n)}(t) = 0$, so $F^{(n)} \ll M$ and f_n is well defined. Now $D_x(t) = \mathbb{E} \sum_{i=1}^\infty X_i I_{\{S_i \leq t\}} = \sum_{i=1}^\infty \int_0^t \mathbb{E}(X_i | S_i = s) dF^{(i)}(s) = \int_0^t \sum_{i=1}^\infty \mathbb{E}(X_i | S_i = s) f_i(s) dM(s)$, and therefore $dD_x(s) / dM(s) = \sum_{i=1}^\infty \mathbb{E}(X_i | S_i = s) f_i(s)$. In the following, denote $\tilde{S}_{j-i} \stackrel{\text{def}}{=} S_j - S_i$. Next,

$$\begin{aligned} \mathbb{E} \sum_{i < j \leq N(t)-1} X_i Y_j &= \sum_{i < j} \mathbb{E} X_i Y_j I_{\{S_j \leq t\}} = \sum_{i < j} \mathbb{E} \mathbb{E} \left(X_i Y_j I_{\{\tilde{S}_{j-i} + S_i \leq t\}} \mid S_i \right) \\ &= \sum_{i < j} \int_{\omega=0}^\infty \mathbb{E} \left(X_i Y_j I_{\{\tilde{S}_{j-i} + \omega \leq t\}} \mid S_i = \omega \right) dF^{(i)}(\omega) \end{aligned}$$

$$\begin{aligned}
 &= \sum_{i < j} \int_{\omega=0}^t \mathbb{E} (X_i | S_i = \omega) \mathbb{E} \left(Y_j I_{\{\tilde{S}_{j-i} + \omega \leq t\}} \right) dF^{(i)}(\omega) \\
 &= \sum_{i < j} \int_{\omega=0}^t \mathbb{E} (X_i | S_i = \omega) \mathbb{E} \mathbb{E} \left(Y_j I_{\{\tilde{S}_{j-i} + \omega \leq t\}} \middle| \tilde{S}_{j-i} \right) dF^{(i)}(\omega) \\
 &= \sum_{i < j} \int_{\omega=0}^t \mathbb{E} (X_i | S_i = \omega) \int_{s=\omega}^{\infty} \mathbb{E} \left(Y_{j-i} I_{\{s \leq t\}} \middle| \tilde{S}_{j-i} = s - \omega \right) dF^{(j-i)}(s - \omega) dF^{(i)}(\omega) \\
 &= \sum_{i < j} \int_{\omega=0}^t \int_{s=\omega}^t \mathbb{E} (X_i | S_i = \omega) \mathbb{E} \left(Y_{j-i} \middle| \tilde{S}_{j-i} = s - \omega \right) dF^{(j-i)}(s - \omega) dF^{(i)}(\omega) \\
 &= \int_{\omega=0}^t \int_{s=\omega}^t \sum_{i=1}^{\infty} \sum_{(j-i)=1}^{\infty} \mathbb{E} (X_i | S_i = \omega) \mathbb{E} \left(Y_{j-i} \middle| \tilde{S}_{j-i} = s - \omega \right) dF^{(j-i)}(s - \omega) dF^{(i)}(\omega) \\
 &= \int_{\omega=0}^t \int_{s=\omega}^t \sum_{i=1}^{\infty} \mathbb{E} (X_i | S_i = \omega) \sum_{k=1}^{\infty} \mathbb{E} \left(Y_k \middle| \tilde{S}_k = s - \omega \right) f_k(s - \omega) dM(s - \omega) f_i(\omega) dM(\omega) \\
 &= \int_{\omega=0}^t \int_{s=\omega}^t \sum_{i=1}^{\infty} \mathbb{E} (X_i | S_i = \omega) f_i(\omega) \sum_{k=1}^{\infty} \mathbb{E} \left(Y_k \middle| \tilde{S}_k = s - \omega \right) f_k(s - \omega) dM(s - \omega) dM(\omega) \\
 &= \int_{\omega=0}^t D_y(t - \omega) \sum_{i=1}^{\infty} \mathbb{E} (X_i | S_i = \omega) f_i(\omega) dM(\omega) = \int_{\omega=0}^t D_y(t - \omega) \frac{dD_x(\omega)}{dM(\omega)} dM(\omega) \\
 &= \int_0^t D_y(t - \omega) dD_x(\omega) = \int_0^t D_x(t - \omega) dD_y(\omega). \quad \square
 \end{aligned}$$

The next result from [Brown and Solomon \(1975\)](#) deals with $r_x(t) \stackrel{\text{def}}{=} D_x(t) - a_x t - \dot{b}_x$:

Lemma 2 (Restatement of [Brown and Solomon, 1975, Lemma 3](#)). For the ordinary case, if $F(\cdot)$ is spread out and μ_3 , λ_1 , and $m_{2,1}$ are finite, then

$$\int_0^{\infty} r_x(t) dt = \ell_x,$$

where ℓ_x is defined in (5). Moreover, $r_x(\cdot)$ is directly Riemann integrable and $\lim_{t \rightarrow \infty} t r_x(t) = 0$.

We can now prove [Theorem 2](#), which is the key to our approximation (6).

Proof of Theorem 2. Since $\mathbb{E} T_1 |X_1| \leq (\mathbb{E} T_1 X_1^2 \mathbb{E} X_1)^{1/2}$, $\mathbb{E} T_1^2 |X_1| \leq (\mathbb{E} T_1 X_1^2 \mathbb{E} X_1^3)^{1/2}$, and $\mathbb{E} |X_1 Y_1| \leq (\mathbb{E} X_1^2 \mathbb{E} X_1^2)^{1/2}$, it holds that $m_{1,1}$, $m_{2,1}$, and $p_{0,1,1}$ are finite. Similarly, $n_{1,1}$ and $n_{2,1}$ are finite. It holds,

$$\begin{aligned}
 \text{Cov}(\dot{R}_x(t), \dot{R}_y(t)) &= \mathbb{E} \dot{R}_{xy}(t) + 2 \mathbb{E} \sum_{i < j \leq N(t)-1} X_i Y_j - \mathbb{E} \dot{R}_x(t) \mathbb{E} \dot{R}_y(t) \\
 &= \mathbb{E} \dot{R}_{xy}(t) + 2 \int_0^t D_x(t - s) dD_y(s) - \mathbb{E} \dot{R}_x(t) \mathbb{E} \dot{R}_y(t), \tag{8}
 \end{aligned}$$

where the second step follows from [Lemma 1](#). We have $\mathbb{E} \dot{R}_{xy}(t) = a_{xy} t + \dot{b}_{xy} + o(1)$ from (3) and it follows from [Theorem 1](#) that

$$\mathbb{E} \dot{R}_x(t) \mathbb{E} \dot{R}_y(t) = a_x a_y t^2 + (a_x \dot{b}_y + a_y \dot{b}_x) t + \dot{b}_x \dot{b}_y + o(1). \tag{9}$$

Now,

$$\int_0^t D_x(t - s) dD_y(s) = \int_0^t r_x(t - s) dD_y(s) + \int_0^t (a_x(t - s) + \dot{b}_x) dD_y(s).$$

By [Lemma 2](#), $r_x(\cdot)$ is directly Riemann integrable. It thus follows from a generalization of the key renewal theorem to renewal-reward processes (see [Brown and Ross, 1972](#)) that $\int_0^t r_x(t - s) dD_y(s) = a_y \ell_x + o(1)$. Next,

$$\int_0^t (a_x(t - s) + \dot{b}_x) dD_y(s) = a_x \int_0^t D_y(s) ds + \dot{b}_x D_y(t) = \dot{b}_x D_y(t) + a_x \int_0^t (r_y(s) + a_y s + \dot{b}_y) ds.$$

Now using [Lemma 2](#) we have

$$\int_0^t D_x(t-s) dD_y(s) = a_x a_y t^2 / 2 + (a_x \dot{b}_y + a_y \dot{b}_x) t + a_x \ell_y + a_y \ell_x + \dot{b}_x \dot{b}_y + o(1). \quad (10)$$

Combining the above into [\(8\)](#) yields the result for the ordinary case.

We now move onto the delayed case. Since $R_x(t) = I_{\{T_0 \leq t\}}(X_0 + \dot{R}_x(t - T_0))$ and similarly for $R_y(t)$,

$$R_x(t) R_y(t) = I_{\{T_0 \leq t\}}(X_0 Y_0 + X_0 \dot{R}_y(t - T_0) + Y_0 \dot{R}_x(t - T_0) + \dot{R}_x(t - T_0) \dot{R}_y(t - T_0)).$$

Now, $\mathbb{E} I_{\{T_0 \leq t\}} X_0 Y_0 = \int_0^t \mathbb{E}(X_0 Y_0 | T_0 = s) dF_0(s) = \mathbb{E} X_0 Y_0 + o(1)$. Next,

$$\begin{aligned} \mathbb{E} I_{\{T_0 \leq t\}} X_0 \dot{R}_y(t - T_0) &= \mathbb{E} I_{\{T_0 \leq t\}} \mathbb{E}(X_0 | T_0) (a_y(t - T_0) + \dot{b}_y + r_y(t - T_0)) \\ &= (a_y t + \dot{b}_y) \mathbb{E} X_0 - a_y \mathbb{E} T_0 X_0 + o(1), \end{aligned}$$

since $r_y(t)$ converges to 0 as $t \rightarrow \infty$ ([Theorem 1](#)) and both $\sup_t \{|r_y(t)|\}$ and $\mathbb{E}|X_0|$ are finite, it holds that $\int_0^t r_y(t-s) \mathbb{E}(X_0 | T_0 = s) dF_0(s) \rightarrow 0$ as $t \rightarrow \infty$. Similarly for $\mathbb{E} I_{\{T_0 \leq t\}} Y_0 \dot{R}_x(t - T_0)$.

Set $\bar{r}(t) \stackrel{\text{def}}{=} \text{Cov}(\dot{R}_x(t), \dot{R}_y(t)) - c_{x,y} t - \dot{d}_{x,y}$. Hence,

$$\begin{aligned} &\mathbb{E} I_{\{T_0 \leq t\}} \dot{R}_x(t - T_0) \dot{R}_y(t - T_0) \\ &= \mathbb{E} I_{\{T_0 \leq t\}} (c_{x,y}(t - T_0) + \dot{d}_{x,y} + \bar{r}(t - T_0) + (a_x(t - T_0) + \dot{b}_x + r_x(t - T_0))(a_y(t - T_0) + \dot{b}_y + r_y(t - T_0))) \\ &= \mathbb{E} I_{\{T_0 \leq t\}} (a_x a_y t^2 + t(c_{x,y} - 2a_x a_y T_0 + \dot{b}_y a_x + \dot{b}_x a_y) + \dot{d}_{x,y} - c_{x,y} T_0 + a_x a_y T_0^2 \\ &\quad + \dot{b}_x \dot{b}_y - \dot{b}_x a_y T_0 - \dot{b}_y a_x T_0 + \bar{r}(t - T_0) + r_x(t - T_0) r_y(t - T_0) \\ &\quad + r_x(t - T_0)(a_y(t - T_0) + \dot{b}_y) + r_y(t - T_0)(a_x(t - T_0) + \dot{b}_x)). \end{aligned}$$

By [Theorem 1](#) and the result proved above for the ordinary case, $r_x(t)$, $r_y(t)$, and $\bar{r}(t)$ all converge to 0. Moreover, $\sup_t \{|\bar{r}(t)|\}$, $\sup_t \{|r_x(t)|\}$, and $\sup_t \{|r_y(t)|\}$ are finite, thus $\int_0^t \bar{r}(t-x) dF_0(x)$, $\int_0^t r_x(t-x) r_y(t-x) dF_0(x)$, $\int_0^t r_x(t-x) dF_0(x)$, and $\int_0^t r_y(t-x) dF_0(x)$ all converge to 0 as $t \rightarrow \infty$. Further, by [Lemma 2](#), $t r_x(t)$ and $t r_y(t)$ also converge to 0, and it easily follows that $\sup_t \{|t r_x(t)|\}$ and $\sup_t \{|t r_y(t)|\}$ are finite; thus

$$\begin{aligned} &\mathbb{E} I_{\{T_0 \leq t\}} (a_x(t - T_0) r_y(t - T_0) + a_y(t - T_0) r_x(t - T_0)) \\ &= \int_0^t (a_x(t - T_0) r_y(t - T_0) + a_y(t - T_0) r_x(t - T_0)) dF_0(x), \end{aligned}$$

which converges to 0 as $t \rightarrow \infty$. Therefore,

$$\begin{aligned} \mathbb{E} I_{\{T_0 \leq t\}} \dot{R}_x(t - T_0) \dot{R}_y(t - T_0) &= a_x a_y t^2 + (c_{x,y} - 2a_x a_y \mathbb{E} T_0 + \dot{b}_y a_x + \dot{b}_x a_y) t + \dot{d}_{x,y} - c_{x,y} \mathbb{E} T_0 \\ &\quad + a_x a_y \mathbb{E} T_0^2 + \dot{b}_x \dot{b}_y - \dot{b}_x a_y \mathbb{E} T_0 - \dot{b}_y a_x \mathbb{E} T_0 + o(1). \end{aligned}$$

Thus,

$$\begin{aligned} \mathbb{E} R_x(t) R_y(t) &= a_x a_y t^2 + (c_{x,y} - 2a_x a_y \mathbb{E} T_0 + \dot{b}_y a_x + \dot{b}_x a_y + a_x \mathbb{E} Y_0 + a_y \mathbb{E} X_0) t \\ &\quad + \dot{d}_{x,y} - c_{x,y} \mathbb{E} T_0 + a_x a_y \mathbb{E} T_0^2 - a_x \mathbb{E} T_0 Y_0 - a_y \mathbb{E} T_0 X_0 + \mathbb{E} X_0 Y_0 \\ &\quad + \dot{b}_x \dot{b}_y - \dot{b}_x a_y \mathbb{E} T_0 - \dot{b}_y a_x \mathbb{E} T_0 + \dot{b}_x \mathbb{E} Y_0 + \dot{b}_y \mathbb{E} X_0 + o(1). \end{aligned}$$

By [Theorem 1](#),

$$\begin{aligned} \mathbb{E} R_x(t) \mathbb{E} R_y(t) &= (a_x t + \dot{b}_x + \mathbb{E} X_0 - a_x \mathbb{E} T_0) (a_y t + \dot{b}_y + \mathbb{E} Y_0 - a_y \mathbb{E} T_0) + o(1) \\ &= a_x a_y t^2 + (a_x \dot{b}_y + a_x \mathbb{E} Y_0 - 2a_x a_y \mathbb{E} T_0 + \dot{b}_x a_y + a_y \mathbb{E} X_0) t \\ &\quad + \dot{b}_x \dot{b}_y + \dot{b}_x \mathbb{E} Y_0 - \dot{b}_x a_y \mathbb{E} T_0 + \dot{b}_y \mathbb{E} X_0 + \mathbb{E} X_0 \mathbb{E} Y_0 - a_y \mathbb{E} X_0 \mathbb{E} T_0 \\ &\quad - a_x \dot{b}_y \mathbb{E} T_0 - a_x \mathbb{E} Y_0 \mathbb{E} T_0 + a_x a_y (\mathbb{E} T_0)^2 + o(1). \end{aligned}$$

Combining the two expressions above yields the result. \square

5. Outlook

The renewal-reward process with multivariate rewards analysed here often plays a role as part of a more complicated stochastic model – for example in multidimensional risk models, such as in [Badila et al. \(2015\)](#). Our results may help analysis of such risk models, at least in some asymptotic regime.

We have allowed the distribution of \mathbf{Z}_n to depend on n in a simple way by allowing \mathbf{Z}_0 to follow a different distribution to $\{\mathbf{Z}_n\}_{n=1}^{\infty}$. A possible extension of our work is to allow for more general dependencies of \mathbf{Z}_n on n by partitioning \mathbb{N} into possibly infinite subsets. Spătaru (2010) gives a CLT for the case of univariate rewards ($L = 1$) with unit rewards (a renewal process) in this setting. Extending Spătaru's result to the univariate or even the multivariate renewal-reward case remains a challenge.

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