Statistics and Probability Letters 104 (2015) 87–93

Contents lists available at ScienceDirect

Statistics and Probability Letters

journal homepage: www.elsevier.com/locate/stapro

A dynamic view to moment matching of truncated distributions

Benoit Liquet*, Yoni Nazarathy

School of Mathematics and Physics, The University of Queensland, St Lucia, Brisbane, 4072 Queensland, Australia

ARTICLE INFO

Article history:
Received 4 May 2015
Received in revised form 7 May 2015
Accepted 7 May 2015
Available online 15 May 2015

Keywords:
Location-scale families
Moment matching
Normal distribution
Truncated distributions

ABSTRACT

We derive an ordinary differential equation (ODE) for solving moment matching problems of truncated univariate distributions. Our method produces a trajectory that solves a family of moment matching problems for different truncation values all with the same target moments.

1. Introduction

The problem of matching distributional parameters to obtain desired moments is almost as old as the field of probability and statistics, see for example the classic book Shohat and Tamarkin (1943). In this paper we put forward an idea for solving the moment matching problem using a novel dynamic method, specifically suited for truncated distributions.

Consider a family of univariate continuous distributions (density functions), $\{g(x;\,\theta),\ \theta\in\Theta\}$, where the parameter space Θ is some subset of $\mathbb{R}^p$. Then, given desired moments, $m_1^*,\ldots,m_p^*$, the moment matching problem aims to find a solution, $\theta\in\Theta$, to the equations,

$$\int x^i g(x;\,\theta)\,dx = m_i^*,\quad i=1,\ldots,p. \tag{1}$$

We consider the case where the density $g(\cdot\,;\,\theta)$ is a truncated distribution of the form,

$$g_{a,b}(x;\,\theta)=\frac{f(x;\,\theta)}{\int_a^b f(u;\,\theta)\,du},$$

where $\{f(x;\,\theta),\ \theta\in\Theta\}$ is a family of densities with support $\mathcal{S}\subset\mathbb{R}$ and $(a,b)\subset S$. Truncated distributions are ubiquitous in probability and statistics. See for example Robert (1995), Arismendi (2013) and Rosenbaum (1961).

We introduce a method to solve the moment matching problem for $g_{a,b}(\cdot\,;\,\cdot)$ by means of an ordinary differential equation (ODE). In fact, our method produces a range of solutions $\theta(z)$ with $z\in[0,1]$, where for each z we have a solution to the moment matching problem (1) associated with $g_{a(z),b(z)}(\cdot\,;\,\cdot)$. Here $\big(a(0),b(0)\big)=\mathcal{S}$, $\big(a(1),b(1)\big)=(a,b)$ and as z

* Corresponding author.
E-mail address: b.liquet@uq.edu.au (B. Liquet).

http://dx.doi.org/10.1016/j.spl.2015.05.006

increases from 0 to 1 the truncation interval $[a(z), b(z)]$ shrinks from $\mathcal{S}$ to the target truncation interval $[a, b]$. We use basic calculus to derive our ODE. The ODE for $\theta(z)$ describes the trajectory within the parameter space Θ such that the desired moments $m_1^*, \ldots, m_p^*$ are maintained throughout.

In practice, solving the moment matching problem when $[a, b] = \mathcal{S}$ is often explicit and immediate, but solving the truncated moment problem does not admit a closed form solution. As an example, consider the exponential distribution, $f(x\,;\,\theta) = \theta e^{-\theta x}\mathbf{1}\{x \in [0, \infty)\}$ where $\mathcal{S} = [0, \infty)$ with $\Theta = (0, \infty)$. In this case given a desired mean, m_1^*, the non-truncated moment matching solution is simply $\theta(0) = 1/m_1^*$, but for a truncation range $[a, b] \subset \mathcal{S}$ one needs to solve the equation (in θ),

$$m_1^* = \theta^{-1}\frac{(b\theta + 1)e^{a\theta} - (a\theta + 1)e^{b\theta}}{e^{a\theta} - e^{b\theta}}, \tag{2}$$

which does not admit a closed form solution. Thus to solve (2) one typically resorts to numerical methods, such as for e.g., Newton's method. Our dynamic (ODE based) approach is fundamentally different because it solves (2) for a whole range of truncation intervals simultaneously.

Besides the exponential distribution (which we handle mostly for the purpose of exposition), we also derive our equations for arbitrary continuous distributions with p parameters. We then specialise to location-scale families ($p = 2$), of which the normal distribution is a special case. Moment matching for truncated normal distributions has been previously studied in the literature, Dyer (1973) and Rosenbaum (1961), but to the best of our knowledge, the dynamic view which we put forward here is novel.

The remainder of the paper is structured as follows. Section 2 illustrates the main idea of the method through the exponential distribution. Section 3 puts forward general equations of our ODE based approach. Section 4 presents the solution for general location scale families and applies this to normal distributions. We conclude and pose open problems in Section 5.

2. Illustration of the main idea through the exponential distribution

To illustrate the main idea of our method, we consider the exponential distribution as presented in the introduction, but focus on the one-sided truncation $[0, b]$ with desired first moment, m_1^*. Note that by comparing to the (extreme case) of a uniform distribution, it is easy to see that it must hold that $m_1^* < b/2$.

Our goal is then to find a solution $\theta > 0$ for the equation,

$$\int_0^b x\frac{f(x\,;\,\theta)}{\int_0^b f(u\,;\,\theta)\,du}\,dx = m_1^*, \quad \text{or alternatively} \quad \int_0^b (x - m_1^*)f(x\,;\,\theta)\,dx = 0,$$

i.e., this is a solution to (2) (with $a = 0$).

We now take $z \in (0, 1]$ and for each z consider the truncation interval, $(0,\ b + (1-z)/z)$. For z close to zero, the interval is close to the non-truncated $[0, \infty)$ support and as z increases towards 1 we have that the interval shrinks to the target interval $[0, b]$. Now consider $\{\theta(z),\ z \in (0, 1]\}$ as a solution to the moment matching problem for every z. That is,

$$\int_0^{b+(1-z)/z} (x - m_1^*)f\big(x\,;\,\theta(z)\big)\,dx = 0, \quad z \in (0, 1].$$

Taking derivative with respect to z and using Leibniz's integral rule we get,

$$\int_0^{b+(1-z)/z} (x - m_1^*)\frac{d}{dz}f\big(x\,;\,\theta(z)\big)\,dx - \frac{1}{z^2}\left(b + \frac{1-z}{z} - m_1^*\right)f\left(b + \frac{1-z}{z}\,;\,\theta(z)\right) = 0. \tag{3}$$

Now observe that,

$$\frac{d}{dz}f\big(x\,;\,\theta(z)\big) = \theta'(z)e^{-\theta(z)x}(1 - \theta(z)x),$$

and plug this derivative into (3). After rearranging, explicitly carrying out the integration and further simplifying we get the ODE:

$$\begin{aligned}\theta'(z) &= \frac{\frac{1}{z^2}(b + \frac{1-z}{z} - m_1^*)\theta(z)e^{-\theta(z)\left(b+\frac{1-z}{z}\right)}}{\int_0^{b+(1-z)/z}(x - m_1^*)e^{-\theta(z)x}(1 - \theta(z)x)\,dx} \\ &= \theta^3(z)\frac{b + \frac{1-z}{z} - m_1^*}{\big((b-1)z + 1\big)\big(z\theta(z)(b - m - 1) + z + \theta(z)\big)\theta(z) + z^2(1 - e^{\theta(z)(b+\frac{1-z}{z})})}.\end{aligned} \tag{4}$$

As an initial condition for the ODE, we take $\theta(0^+) = \frac{1}{m_1^*}$ since this is the solution of the non-truncated moment matching problem.

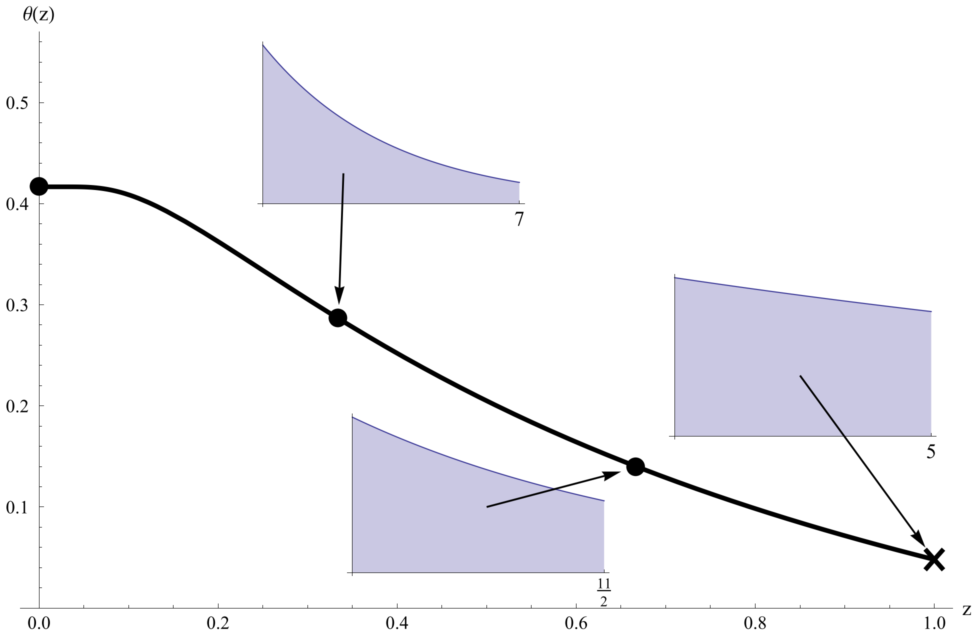

Fig. 1. The trajectory of $\theta(z)$ for the exponential distribution with $m_1^* = 2.4$, $b = 5$ and $\epsilon = 0.01$.

In practice, to use the method, we solve the ODE numerically using some arbitrary ODE solver on $z \in [\epsilon, 1]$ where ϵ is chosen small such that $\theta(\epsilon) \approx \frac{1}{m_1^*}$. We illustrate this for the case of $m_1^* = 2.4$ and $b = 5$. We choose $\epsilon = 0.01$. Note that in this case, $b + (1-\epsilon)/\epsilon = 104$. Fig. 1 presents the trajectory as well as several truncated distributions along the way: at $z = 1/3$ the truncation interval is $[0, 7]$ and $\theta(z) = 0.28701$, at $z = 2/3$ the truncation interval is $[0, 5.5]$ and $\theta(z) = 0.140213$, then finally at $z = 1$ the truncation interval is $[0, 5]$ and $\theta(z) = 0.0480456$. Note that the absolute error (the absolute difference between the left hand side and the right hand side of Eq. (2) with $a = 0$) throughout the trajectory is bounded by 1.5×10^{-6}.

3. The general ODE

Having illustrated the method for the exponential distribution, we now put forward general equations for univariate distributions with support $\mathbb{R}$, p parameters and an equal number of desired moments. As input to our method we are given: (i) A distributional family with densities, $\{f(x;\,\theta),\ \theta \in \Theta\}$ with $\Theta \subset \mathbb{R}^p$ and support $\mathbb{R}$ where we assume that $f(x;\,\theta_1,\ldots,\theta_p)$ is differentiable with respect to each θ_i; (ii) Desired moments $m_1^*,\ldots,m_p^*$; and (iii) A target interval $[a, b]$ (possibility with $a = -\infty$ or $b = \infty$).

A goal is to find a solution $\theta \in \Theta$ (if one exists) for the equations,

$$\int_a^b x^i \frac{f(x;\,\theta)}{\int_a^b f(u;\,\theta)\,du}\,dx = m_i^*, \quad i = 1,\ldots,p.$$

or alternatively,

$$\int_a^b (x^i - m_i^*) f(x;\,\theta)\,dx = 0, \quad i = 1,\ldots,p. \tag{5}$$

We now take $z \in (0, 1]$ and for each z consider the truncation interval,

$$(l_z, h_z) = \left(a - \frac{1-z}{z},\ b + \frac{1-z}{z}\right).$$

Thus as $z \to 0$ we have $(l_z, h_z) \to (-\infty, \infty)$ and as z increases up to 1, the interval shrinks to (a, b).

Remark. We note that the choice of the functions (l_z, h_z) is quite modular and may also depend on the support of the distributional family and the type of truncation interval (single sided, two sided, etc.).

Now consider, $\{\theta(z),\ z \in (0, 1]\}$ and write a dynamic version of (5) as,

$$\int_{l_z}^{h_z} (x^i - m_i^*) f(x;\,\theta(z))\,dx = 0, \quad i = 1,\ldots,p.$$

Taking derivative with respect to z and applying Leibniz's integral rule we get,

$$\int_{l_z}^{h_z} (x^i - m_i^*) \frac{d}{dz} f\big(x\,;\,\theta(z)\big)\,dx + (h_z^i - m_i^*) f\big(h_z\,;\,\theta(z)\big)\frac{d}{dz}h_z - (l_z^i - m_i^*) f\big(l_z\,;\,\theta(z)\big)\frac{d}{dz}l_z = 0.$$

Defining,

$$c_i(z, \theta(z)) = (h_z^i - m_i^*) f\big(h_z\,;\,\theta(z)\big) + (l_z^i - m_i^*) f\big(l_z\,;\,\theta(z)\big), \quad i = 1, \ldots, p,$$

this can be written as,

$$\int_{l_z}^{h_z} (x^i - m_i^*) \frac{d}{dz} f(x\,;\,\theta(z))\,dx = \frac{1}{z^2} c_i(z, \theta(z)). \tag{6}$$

Using the (multivariate)-chain rule we have,

$$\frac{d}{dz} f\big(x\,;\,\theta(z)\big) = \sum_{j=1}^{p} a_j\big(x, \theta(z)\big)\,\theta_j'(z), \tag{7}$$

where,

$$a_j\big(x, (\widetilde{\theta}_1, \ldots, \widetilde{\theta}_p)\big) = \frac{d}{d\widetilde{\theta}_j} f(x\,;\,\widetilde{\theta}_1, \ldots, \widetilde{\theta}_p), \quad \text{and} \quad \theta_j'(z) = \frac{d}{dz}\theta_j(z).$$

Plugging (7) into (6) and defining,

$$B_{i,j}(z, \theta(z)) = \int_{l_z}^{h_z} (x^i - m_i^*) a_j\big(x, \theta(z)\big)\,dx, \quad i, j = 1, \ldots, p, \tag{8}$$

we get,

$$\sum_{j=1}^{p} B_{i,j}\big(z, \theta(z)\big)\theta_j'(z) = \frac{1}{z^2} c_i\big(z, \theta(z)\big), \quad i = 1, \ldots, p.$$

Now define $B\big(z, \theta(z)\big)$ as the $p \times p$ matrix of $B_{i,j}\big(z, \theta(z)\big)$, treat $\theta(z)$, $\theta'(z)$ and $c\big(z, \theta(z)\big)$ as column vectors. This yields,

$$B\big(z, \theta(z)\big)\,\theta'(z) = \frac{1}{z^2} c\big(z, \theta(z)\big).$$

Our method relies on the assumption that for all $z \in (0, 1]$ the matrix is non-singular (see discussion in Section 5). In this case we get the desired ODE:

$$\frac{d}{dz}\theta(z) = \frac{1}{z^2} B\big(z, \theta(z)\big)^{-1} c\big(z, \theta(z)\big) := F\big(z, \theta(z)\big), \quad z \in (0, 1]. \tag{9}$$

Note that when a closed form of the integrals in (8) is available, $F(\cdot, \cdot)$ can be represented in an explicit form. This is essentially done for the exponential distribution in (4), and further as we show in the next section, simplification of $F(\cdot, \cdot)$ occurs for location-scale families.

As an initial condition for the ODE (9), we take $\theta(0^+)$ as the solution of the non-truncated moment matching problem, i.e., the solution of (5) when $a = -\infty$ and $b = \infty$.

4. The ODE for location scale families

We now consider location-scale families:

$$f(x\,;\,\theta_1, \theta_2) = \frac{1}{\theta_2}\varphi\Big(\frac{x - \theta_1}{\theta_2}\Big), \tag{10}$$

where $\varphi(\cdot)$ is a symmetric density function and $(\theta_1,\ \theta_2) \in \mathbb{R} \times \mathbb{R}_+^*$.

In this case, denoting $\varphi'(u) = \frac{d}{du}\varphi(u)$, the coefficients of the multivariate chain rule as in (7) are,

$$a_1\big(x,\ (\widetilde{\theta}_1, \widetilde{\theta}_2)\big) = -\frac{\varphi'\left(\frac{x-\widetilde{\theta}_1}{\widetilde{\theta}_2}\right)}{\widetilde{\theta}_2^2}, \qquad a_2\big(x,\ (\widetilde{\theta}_1, \widetilde{\theta}_2)\big) = -\frac{\widetilde{\theta}_2\varphi\left(\frac{x-\widetilde{\theta}_1}{\widetilde{\theta}_2}\right) + (x - \widetilde{\theta}_1)\varphi'\left(\frac{x-\widetilde{\theta}_1}{\widetilde{\theta}_2}\right)}{\widetilde{\theta}_2^3}.$$

As an aid, denote for integer $i \geq 0$,

$$n_i = \int_{l_z}^{h_z} x^i f(x;\,\theta)\,dx, \quad \text{and} \quad m_i = \frac{n_i}{n_0}.$$

That is, m_i is the i'th moment under the dynamic truncation interval $[l_z, h_z]$. It is further useful to have for integer $i \geq 0$,

$$p_i = \int_{l_z}^{h_z} x^i \varphi'\left(\frac{x-\theta_1}{\theta_2}\right) dx = \theta_2^2 \left[x^i f(x;\,\theta)\right]_{l_z}^{h_z} - \theta_2^2 \int_{l_z}^{h_z} i x^{i-1} f(x;\,\theta)\,dx = \theta_2^2 \left(\widetilde{h}_i - \widetilde{l}_i - i\, n_{i-1}\right), \tag{11}$$

where we also define for integer $i \geq 0$,

$$\widetilde{l}_i = l_z^i f\big(l_z\,;\,\theta(z)\big), \qquad \widetilde{h}_i = h_z^i f\big(h_z\,;\,\theta(z)\big).$$

We now compute the four $B_{i,j}$ expressions. In doing so, we exploit the fact that on $\theta(z)$, $m_i(z) = m_i^*$ for $i = 1, 2$. This is because $\theta(z)$ is assumed to be such that the desired moments are maintained for any truncation level (any z). Hence,

$$n_i = n_0 m_i = n_0 m_i^*, \quad \text{for } i = 1, 2.$$

Now some tedious, yet straight forward computations yield:

$$\begin{aligned}
B_{1,1} &= -\theta_2^{-2} \int_{l_z}^{h_z} (x - m_1^*)\varphi'\left(\frac{x-\theta_1}{\theta_2}\right) dx \\
&= -\theta_2^{-2}\big(p_1 - m_1^* p_0\big), \\
B_{1,2} &= -\theta_2^{-3} \int_{l_z}^{h_z} (x - m_1^*)\left(\theta_2 \varphi\left(\frac{x-\theta_1}{\theta_2}\right) + (x-\theta_1)\varphi'\left(\frac{x-\theta_1}{\theta_2}\right)\right) dx \\
&= -\theta_2^{-1} \int_{l_z}^{h_z} (x - m_1^*) f(x;\,\theta)\,dx - \theta_2^{-3} \int_{l_z}^{h_z} (x^2 - (\theta_1 + m_1^*)x + m_1^*\theta_1)\varphi'\left(\frac{x-\theta_1}{\theta_2}\right) dx \\
&= -\theta_2^{-1}(n_1 - m_1^* n_0) - \theta_2^{-3}\big(p_2 - (\theta_1 + m_1^*)p_1 + m_1^*\theta_1 p_0\big) \\
&= -\theta_2^{-3}\big(p_2 - (\theta_1 + m_1^*)p_1 + m_1^*\theta_1 p_0\big), \\
B_{2,1} &= -\theta_2^{-2} \int_{l_z}^{h_z} (x^2 - m_2^*)\varphi'\left(\frac{x-\theta_1}{\theta_2}\right) dx \\
&= -\theta_2^{-2}\big(p_2 - m_2^* p_0\big), \\
B_{2,2} &= -\theta_2^{-3} \int_{l_z}^{h_z} (x^2 - m_2^*)\left(\theta_2 \varphi\left(\frac{x-\theta_1}{\theta_2}\right) + (x-\theta_1)\varphi'\left(\frac{x-\theta_1}{\theta_2}\right)\right) dx \\
&= -\theta_2^{-1} \int_{l_z}^{h_z} (x^2 - m_2^*) f(x;\,\theta)\,dx - \theta_2^{-3} \int_{l_z}^{h_z} (x^3 - \theta_1 x^2 - m_2^* x + m_2^*\theta_1)\varphi'\left(\frac{x-\theta_1}{\theta_2}\right) dx \\
&= -\theta_2^{-1}(n_2 - m_2^* n_0) - \theta_2^{-3}\big(p_3 - \theta_1 p_2 - m_2^* p_1 + m_2^*\theta_1 p_0\big) \\
&= -\theta_2^{-3}\big(p_3 - \theta_1 p_2 - m_2^* p_1 + m_2^*\theta_1 p_0\big).
\end{aligned}$$

Using the above expressions the ODE associated with (9) has right hand side as follows:

$$\begin{aligned}
\begin{bmatrix} F_1 \\ F_2 \end{bmatrix} &= \frac{1}{z^2}\frac{1}{\Delta} \begin{bmatrix} B_{2,2} & -B_{1,2} \\ -B_{2,1} & B_{1,1} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} \\
&= \frac{1}{z^2}\frac{1}{\theta_2^3 \Delta} \begin{bmatrix} c_2\left(p_0 m_1^* \theta_1 - p_1\left(m_1^* + \theta_1\right) + p_2\right) - c_1\left(\left(p_0 m_2^* - p_2\right)\theta_1 - p_1 m_2^* + p_3\right) \\ \theta_2\Big(c_2\left(p_0 m_1^* - p_1\right) - c_1\left(p_0 m_2^* - p_2\right)\Big) \end{bmatrix},
\end{aligned}$$

with the determinant,

$$\begin{aligned}
\Delta &= B_{1,1}B_{2,2} - B_{1,2}B_{2,1} \\
&= \frac{p_2\left(p_1 m_1^* + p_0 m_2^*\right) - p_1^2 m_2^* - p_0 p_3 m_1^* + p_3 p_1 - p_2^2}{{\theta_2}^5}.
\end{aligned}$$

As is evident from this form of $F(\cdot,\cdot)$, the right hand side of the ODE can be easily evaluated using,

$$p_0 = \theta_2^2(\widetilde{h}_0 - \widetilde{l}_0), \qquad p_1 = \theta_2^2(\widetilde{h}_1 - \widetilde{l}_1 - n_0), \qquad p_2 = \theta_2^2(\widetilde{h}_2 - \widetilde{l}_2 - n_0 m_1^*), \qquad p_3 = \theta_2^2(\widetilde{h}_3 - \widetilde{l}_3 - n_0 m_2^*).$$

and,

$$c_i = \widetilde{h}_i + \widetilde{l}_i - m_i^*(\widetilde{h}_0 + \widetilde{l}_0),$$

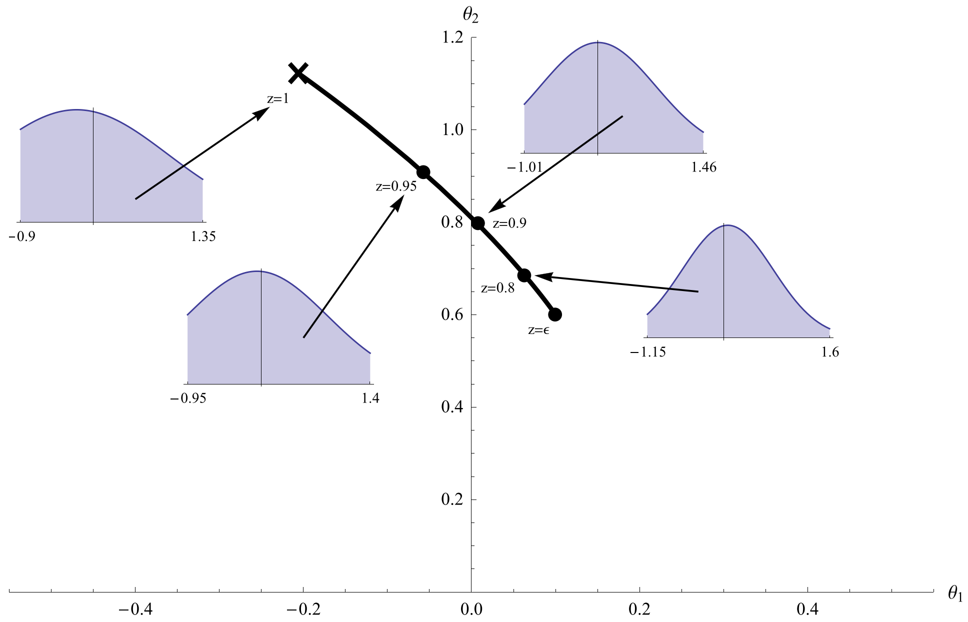

Fig. 2. The trajectory of $\big(\theta_1(z),\ \theta_2(z)\big)$ for the Normal distribution with $m_1^* = 0.1$, $\sqrt{m_2^* - (m_1^*)^2} = 0.6$, $[a, b] = [-0.9, 1.35]$ and $\epsilon = 0.01$.

for any z, where the only potentially non-trivial component is n_0. For example in the case of a Normal distribution (as presented in the example that follows),

$$n_0 = \int_{l_z}^{h_z} \frac{1}{\theta_2(z)}\phi\Big(\frac{x-\theta_1(z)}{\theta_2(z)}\Big)\,dx = \Phi\Big(\frac{h_z-\theta_1(z)}{\theta_2(z)}\Big) - \Phi\Big(\frac{l_z-\theta_1(z)}{\theta_2(z)}\Big),$$

where $\phi(\cdot)$ and $\Phi(\cdot)$ are the standard normal density and CDF respectively.

Illustration for the normal distribution

We now consider the normal distribution as a location-scale example, taking $\varphi(\cdot)$ of (10) to be the standard normal density, $\phi(\cdot)$. Here $\theta_1 \in \mathbb{R}$ is the mean and $\theta_2 > 0$ the standard deviation. If we are given a single desired truncation interval $[a, b]$ and desired moments m_1^* and m_2^*, then the goal is to solve these two equations with two unknowns (based on moment calculations of the truncated normal distribution):

$$\begin{cases} m_1^* = \theta_1 - \theta_2 \dfrac{\phi\left(\frac{b-\theta_1}{\theta_2}\right) - \phi\left(\frac{a-\theta_1}{\theta_2}\right)}{\Phi\left(\frac{b-\theta_1}{\theta_2}\right) - \Phi\left(\frac{a-\theta_1}{\theta_2}\right)}, \\[2ex] m_2^* = \theta_1^2 + \theta_2^2 - \theta_2 \dfrac{(\theta_1+b)\phi\left(\frac{b-\theta_1}{\theta_2}\right) - (\theta_1+a)\phi\left(\frac{a-\theta_1}{\theta_2}\right)}{\Phi\left(\frac{b-\theta_1}{\theta_2}\right) - \Phi\left(\frac{a-\theta_1}{\theta_2}\right)}. \end{cases} \tag{12}$$

These moment matching equations can potentially be solved numerically (e.g., using Newton's method). The virtue of our dynamic method is that we reach a solution for a whole range of truncation intervals ending with the desired interval $[a, b]$.

As an illustration consider $[a, b] = [-0.9, 1.35]$ with desired mean and standard deviation,

$$m_1^* = 0.1, \qquad \sqrt{m_2^* - (m_1^*)^2} = 0.6.$$

Solving the system of ODEs using $\epsilon = 0.01$ we obtain the trajectory in the two parameter space as appearing in Fig. 2. The figure presents the two dimensional trajectory as well as several truncated distributions along the way: at $z = 0.8$ the truncation interval is $[-1.15, 1.6]$, at $z = 0.9$ the truncation interval is $[-1.01, 1.46]$, at $z = 0.95$ the truncation interval is $[-0.95, 1.4]$ and finally at $z = 1$ the truncation interval is $[a, b]$. The trajectory starts with $z = \epsilon$ at the desired mean and standard deviation. Then as z increases and the truncation interval decreases towards $[a, b]$ the trajectory migrates towards,

$$\Big(\theta_1(1),\ \theta_2(1)\Big) = \Big(-0.20606,\ 1.12264\Big).$$

Note that throughout this trajectory the maximum absolute error in both $\theta_1(\cdot)$ and $\theta_2(\cdot)$ is bounded by 1.5×10^{-6}.

Remark. Observe that for the standard normal density, $\phi'(x) = -x\phi(x)$. This can be exploited in the calculation of p_i of (11) to get $p_i = n_0(\theta_1 m_i - m_{i+1})$. Then, this form of p_i can potentially be incorporated into the $B_{i,j}$ calculation so as to get an alternative form to the ODE based on the first four truncated moment (observe p_3 in $B_{2,2}$). This relationship is also directly related to a recursive formula for the truncated normal moments:

$$m_{i+1} = \theta_1 m_i + i\theta_2^2 m_{i-1} - \theta_2 \frac{b^i \phi\left(\frac{b-\theta_1}{\theta_2}\right) - a^i \phi\left(\frac{a-\theta_1}{\theta_2}\right)}{\Phi\left(\frac{b-\theta_1}{\theta_2}\right) - \Phi\left(\frac{a-\theta_1}{\theta_2}\right)}, \quad i = 0, 1, 2, \dots, \tag{13}$$

see Wikipedia (2014) for the details of (13).

5. Concluding remarks

In this paper we put forward an idea for solving moment matching equations by means of a dynamic method. As opposed to generic solution methods of the moment matching equations for a fixed truncation interval, our method produces a continuum of solutions for a range of truncation intervals.

Our exposition in this paper presents the overarching idea, yet there remain technical issues which require further investigation. For example, since our equations are defined only for $z > 0$, our method starts with $z = \epsilon$ small. It remains to be verified that as $\epsilon \to 0$ the trajectory of the ODE converges to the exact moment matching trajectory. This stability property seems apparent from our numerical experiments, yet we have still not proved it. A further question deals with the non-singularity assumption of the matrix $B(\cdot, \cdot)$. We still do not have explicit conditions for this.

A more fundamental issue that requires investigation deals with existence and uniqueness of solutions to the truncated moment matching problem and the relationship of the ODE trajectory to this. For example, ideally we would want that if $\theta(z)$ hits the boundary of the parameter space Θ for some $z < 1$, that it would imply non-existence of a solution (such is for example the case for the exponential distribution, see Fig. 2). Yet, in generality this property of our ODE seems hard to verify. For example, even in the case of a location-scale normal distribution, to the best of our knowledge, there do not exist explicit results about regions where the solution of (12) exists and/or is unique.

In addition to the above technicalities, our novel method can be extended and refined. Extending it to the case of multi-variate distributions, truncated within rectangular boxes, can be done using the same principles that we used. Yet in this case, manipulating the expressions in hand to produce a useable ODE appears to be a challenging task. A further challenge which perhaps requires application of some elementary differential geometry is to consider non-rectangular regions in the multi-variate setting (see for example Flanders (1973) for discussion of such a generalisation of Leibniz's rule).

Finally, an applicative goal is to optimise the numerical algorithms of solving the associated ODE so as to minimise error and maximise computational efficiency.

Acknowledgements

We would like to thank Ross McVinish, Peter Taylor and Eric Orjebin for useful comments. YN is supported by Australian Research Council (ARC) grants DP130100156 and DE130100291.

References

Arismendi, J.C., 2013. Multivariate truncated moments. J. Multivariate Anal. 117, 41–75.
Dyer, D.D., 1973. On moments estimation of the parameters of a truncated bivariate normal distribution. Appl. Stat. 287–291.
Flanders, H., 1973. Differentiation under the integral sign. Amer. Math. Monthly 615–627.
Robert, C.P., 1995. Simulation of truncated normal variables. Stat. Comput. 5 (2), 121–125.
Rosenbaum, S., 1961. Moments of a truncated bivariate normal distribution. J. R. Stat. Soc. Ser. B Stat. Methodol. 405–408.
Shohat, J.A., Tamarkin, J.D., 1943. The Problem of Moments, Vol. 1. American Mathematical Society.
Wikipedia, Truncated normal distribution: a recursive formula, 2014. [Online; accessed 03.05.15].