Queueing Syst (2011) 68:385–394
DOI 10.1007/s11134-011-9249-7

The variance of departure processes: puzzling behavior and open problems

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Received: 9 May 2011 / Revised: 9 May 2011 / Published online: 17 August 2011

Abstract We consider the variability of queueing departure processes. Previous results have shown the so-called BRAVO effect occurring in M/M/1/K and GI/G/1 queues: Balancing Reduces Asymptotic Variance of Outputs. A factor of $(1 - 2/\pi)$ appears in GI/G/1 and a factor of $1/3$ appears in M/M/1/K, for large K. A missing piece in the puzzle is the GI/G/1/K queue: Is there a BRAVO effect? If so, what is the variability? Does $1/3$ play a role?

This open problem paper addresses these questions by means of numeric and simulation results. We conjecture that at least for the case of light tailed distributions, the variability parameter is $1/3$ multiplied by the sum of the squared coefficients of variations of the inter-arrival and service times.

Keywords Queueing theory · Loss systems · Asymptotic variance rate · BRAVO

Mathematics Subject Classification (2000) 60J27 · 60K25

1 Introduction

Departure processes of queues have received considerable attention in the literature. A classic result is Burke's theorem from the 1950s, stating that departures of the stationary M/M/1 follow a Poisson process [6]. Following this result, the 1960s and 1970s have witnessed dozens of studies, analyzing various attributes of the departure stream. Quite comprehensive surveys are in [8] and [11]. During the 1980s and 1990s, more research has focused on departure processes, mostly due to the emergence of

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queueing network decomposition schemes [16]. In addition, a variety of studies taking the viewpoint of manufacturing, have analyzed the output variance of stochastic production lines; cf. [15] and references therein.

Some more recent studies ([2, 12] and [13]) have brought back attention to elementary models. In this respect, the asymptotics of the variance curve have been analyzed. Rather surprising results have shown that when the service capacity is set to match the arrival rate, there is a significant decrease in the departure variance. This type of phenomena has been termed BRAVO (Balancing Reduces Asymptotic Variance of Outputs). It was first observed for M/M/1/K in [13], and has recently been established for GI/G/1 queues under quite general conditions [2].

The purpose of this note is to augment the known BRAVO results by presenting numerical and simulation results for the GI/G/1/K queue (with K finite). In this respect, we formulate a conjecture and highlight some of the puzzling behavior and open problems.

The GI/G/1/K queue is a single server queue operating under a work conserving service discipline, having a renewal arrival stream and i.i.d. service times. There are $K-1$ waiting positions, customers who arrive at a full system do not enter and leave for ever. We assume finite second moments of the inter-arrival and service times with means and squared coefficient of variations (the variance divided by the mean squared), λ^{-1}, μ^{-1}, c_a^2 and c_s^2, respectively. In the cases where a stationary distribution exists, we may assume the system is stationary, otherwise, begin with an empty system. In the discussion below we will also treat cases where $K=\infty$ (no buffer limit) and in these cases denote the corresponding systems using the usual notation (i.e., M/M/1, GI/G/1, etc.).

Denote the departure process by $D(t)$. This is a count of the number of completed services during $[0,t]$. The following basic quantities are typically of interest:

$$\lambda^* = \lim_{t\to\infty} \frac{\mathbb{E}[D(t)]}{t}, \qquad v^* = \lim_{t\to\infty} \frac{\mathrm{Var}[D(t)]}{t}, \quad \gamma^* = \frac{v^*}{\lambda^*}.$$

We shall assume throughout that these limits exist and are finite (see [10] for an example of some GI/G/1 queues where v^* is not finite). We refer to v^* as the *asymptotic variance*. The ratio γ^* is sometimes referred to as the index of dispersion of counts (cf. [7] or [9]), in this paper we simplify refer to it as the *variability parameter*.[1] It is a rough measure of the long term variability of the point process. The variability parameter of a Poisson process is 1 and sometimes serves as a reference point. More generally, the variability parameter of a renewal process equals the squared coefficient of variation of the inter-renewal times.

It is quite straightforward to establish that

$$\lambda^* = \min\{\lambda,\mu\} + o_K(1),$$

where $o_K(1)$ vanishes as $K\to\infty$. It is further quite evident that

$$v^* = \begin{cases} \lambda c_a^2 + o_K(1) & \text{if } \lambda < \mu, \\ \mu c_s^2 + o_K(1) & \text{if } \lambda > \mu, \end{cases} \quad \text{and thus}$$

[1] In our simulation results we shall also plot the transient $\gamma^*(t) = \mathrm{Var}(D(t))/\mathbb{E}[D(t)]$.

$$\gamma^* = \begin{cases} c_a^2 + o_K(1) & \text{if } \lambda < \mu, \\ c_s^2 + o_K(1) & \text{if } \lambda > \mu. \end{cases}$$

To argue the above heuristically, consider first the case where $K = \infty$. If $\lambda < \mu$, the queue is stable and thus the departure process is a random translation of the renewal arrival process. It is thus easy to show that the departure process possesses many of the asymptotic characteristics of the arrival process (cf. [17, Theorem 1.1]). In the $\lambda > \mu$ case, the argument is different: after a finite time, the system remains non-empty for ever and from this time on, the departures are a renewal process of the services. For finite K, the same approximately holds for $\lambda \ll \mu$ or $\lambda \gg \mu$ and holds asymptotically as $K \to \infty$, when $\lambda \neq \mu$.[2]

Summarizing the above, we see that for $\lambda \neq \mu$, the variability parameter is either determined by the arrival or by the service process, but not both (when $K < \infty$, this is approximately true, i.e., up to the $o_K(1)$ term). For the critically loaded case, one may expect both the arrival and service processes to play a role. Perhaps the most straightforward guess is:

$$\text{(Wrong guess)} \quad \gamma^* = \frac{1}{2}\left(c_a^2 + c_s^2\right) + o_K(1). \tag{1}$$

This is especially sensible in the M/M/1 or M/M/1/K queues, as it would imply that $\gamma^* = 1$ everywhere. Quite surprisingly (1) is not the correct guess. The following has been shown in [13] for M/M/1/K and related birth death queues and later in [2] for the M/M/1:

$$\gamma^* = \begin{cases} 2(1 - \frac{2}{\pi}) & \text{for M/M/1 with } \lambda = \mu, \\ \frac{2}{3} - \frac{1}{3K} + \frac{1}{3K(K+1)} & \text{for M/M/1/K with } \lambda = \mu. \end{cases} \tag{2}$$

Thus in the critical M/M/1, the variability parameter is approximately 0.72 and in the M/M/1/K it is approximately $2/3$ for large K. Put differently, the naive guess (1) should be reduced by a factor of either 0.72 or $2/3$ depending on the model, infinite buffer or finite buffer. It has been further established in [2] that for a wide class of GI/G/1 systems with $\lambda = \mu$:

$$\text{(GI/G/1)} \quad \gamma^* = \left(c_a^2 + c_s^2\right)\left(1 - \frac{2}{\pi}\right). \tag{3}$$

The case that is still an open problem is GI/G/1/K. Combining the known results (2) and (3), a sensible guess is:

$$\text{(Conjecture for GI/G/1/K)} \quad \gamma^* = \left(c_a^2 + c_s^2\right)\frac{1}{3} + o_K(1). \tag{4}$$

[2]Making the arguments of this paragraph precise is quite simple for the $\lambda < \mu$ case, yet it is not clear what are the minimal assumptions that are required of the initial distribution and inter-arrival time and service moments. For the $\lambda > \mu$ case, it is less trivial, since the variance of the number of departures up to the final finite busy period may be infinite in certain cases. We are not aware of a proof.

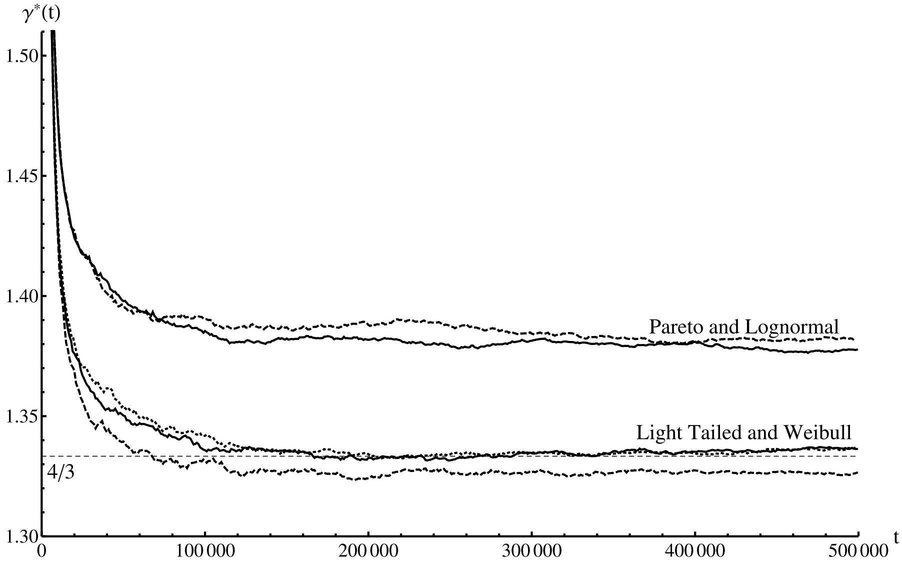

Fig. 1 Simulation estimates of $\gamma^*(t)$ for GI/G/1/200 queues with $\lambda = \mu = 1$ and $c_a^2 = c_s^2 = 2$. The *top curves* are of systems driven by either Pareto (*solid curve*) or Lognormal (*dashed curve*) distributions. These do not appear to converge to 4/3. The *bottom curves* are of systems driven by either Bimodal distributions (*dashed curve*), phase-type distributions (*dotted curve*) or heavy-tailed Weibull distributions (*solid curve*). These appear to converge to the conjectured value

Our main contribution is to supply supporting numerical evidence which indicates that (4) is indeed correct, at least for light-tailed inter-arrival and service distributions.[3] We are not sure about heavy-tailed distributions or about systems that incorporate both light and heavy tails.

The remainder of the paper is organized as follows: Sect. 2 states our conjecture and highlights some puzzling subtleties. Section 3 outlines the supporting numerical computations that we have done for PH/PH/1/K example cases. Section 4 outlines details regarding simulation runs that we have performed. We conclude in Sect. 5.

2 A conjecture

The following conjecture constitutes the main open problem of this paper:

Conjecture 1 *For the GI/G/1/K queue with $\lambda = \mu$ and light-tailed inter-arrival and service times*,

$$\gamma^* = \frac{1}{3}\left(c_a^2 + c_s^2\right) + O\left(\frac{1}{K}\right).$$

[3]We refer to a distribution of a random variable X as *light-tailed* if $\mathbb{E}[e^{X\theta}] < \infty$ for some $\theta > 0$, otherwise we denote it as *heavy-tailed*.

Note that while the above conjecture is formulated for light-tailed service times, our simulations indicate that this is not a necessary condition. Figure 1 presents estimates of $\gamma^*(t)$ for some GI/G/1/200 systems with $c_a^2 = c_s^2 = 2$. The figure indicates that Lognormal and Pareto distributions exhibit one type of behavior (having a BRAVO effect but possibly not with a factor of $1/3$) and in contrast the heavy-tailed Weibull exhibits behaviors similar to light-tailed distributions as in Conjecture 1. We are thus quite confident that the exponential moments condition of our conjecture is too strong, yet we are not sure what the minimal condition is.

A first guess (paralleling the GI/G/1 analysis of [2]) is the existence of $2 + \epsilon$ moments, for $\epsilon > 0$, but the simulation results indicate differently since this condition is met by all the distributions which we chose. Note also that it is possible that for the Lognormal and the Pareto distributions which we simulate, either the convergence of $\gamma^*(t)$ to γ^* is very slow as $t \to \infty$, or perhaps the $o_K(1)$ term which we conjecture to be $O(1/K)$ for light-tailed distributions, vanishes at a much slower rate as $K \to \infty$, with these distributions.

Further details of the simulation are presented in Sect. 4.

3 Numerical computations for PH/PH/1/K examples

PH/PH/1/K queues are GI/G/1/K queues with phase-type distributions (cf. [5]). The variability parameter of some special cases with $c_a^2 = c_s^2$ has already been considered in [13] and presented in Fig. 9 of that paper. Those results already hint that Conjecture 1 holds when $c_a^2 = c_s^2$. We now extend the computations to cases where $c_a^2 \neq c_s^2$ using a similar framework. The analysis uses a matrix-analytic framework to represent the departure process of PH/PH/1/K queues as a MAP (Markov Arrival Process). We do not repeat the technicalities of MAPs, but rather point the reader to Sect. 2 of [13] and more generally to Chap. XI, Sect. 1a of [3].

Our computations are for PH/PH/1/K queues with the inter-arrival and service time distributions having mean 1 and parameterized by their squared coefficient of variation, which we denote by c^2. In the case of $0 < c^2 < 1$, we use a distribution that is a sum of $n = \lceil \frac{1}{c^2} \rceil$ independent exponential random variables. The first random variable has mean μ_1^{-1}, the remaining $n-1$ random variables have mean μ_2^{-1} with

$$\mu_1 = \frac{n}{1 + \sqrt{(n-1)(nc^2 - 1)}}, \qquad \mu_2 = \mu_1 \frac{n-1}{\mu_1 - 1}.$$

In the case of $c^2 \geq 1$, we use a hyper-exponential distribution (a mixture of two exponential random variables): With probability $p = (c-1)/(c+1)$, it is an exponential random variable having mean 2. With probability $1 - p$ it is exponential having mean $(1-2p)/(1-p)$.

Table 1 contains results evaluated for 16 systems with various c_a^2 and c_s^2. Observe that as K grows the results converge to $\frac{1}{3}(c_a^2 + c_s^2)$ which is presented in the right most column. To understand the computation involved in generating this table, consider for example the third row, (0.15, 1.5). In this case, the inter-arrival distribution is a sum of 7 phases and the service distribution is hyper-exponential with 2 phases. Thus the

Table 1 Values of γ^*, of some PH/PH/1/K queues for increasing K. The right most column is for $\lim_{K\to\infty}\gamma^*$ as in Conjecture 1, observe that it matches the column of $K=100$ quite well

(c_a^2, c_s^2)	$K=10$	$K=20$	$K=30$	$K=40$	$K=50$	$K=75$	$K=100$	$\frac{1}{3}(c_a^2+c_s^2)$
(0.15, 0.15)	0.100	0.100	0.100	0.100	0.100	0.100	0.100	1/10
(0.15, 1.00)	0.379	0.381	0.382	0.382	0.382	0.383	0.383	23/60
(0.15, 1.50)	0.576	0.564	0.559	0.557	0.556	0.554	0.553	11/20
(0.25, 0.80)	0.348	0.349	0.349	0.349	0.349	0.350	0.350	7/20
(0.40, 0.20)	0.199	0.200	0.200	0.200	0.200	0.200	0.200	1/5
(0.50, 0.50)	0.325	0.329	0.331	0.331	0.332	0.332	0.333	1/3
(0.60, 1.00)	0.519	0.526	0.528	0.530	0.530	0.531	0.532	8/15
(0.75, 0.75)	0.493	0.497	0.498	0.498	0.499	0.499	0.499	1/2
(0.75, 1.50)	0.770	0.763	0.759	0.757	0.756	0.754	0.753	3/4
(1.00, 0.15)	0.378	0.381	0.382	0.382	0.382	0.383	0.383	23/60
(1.00, 1.00)	0.636	0.651	0.656	0.659	0.660	0.662	0.663	2/3
(1.20, 0.60)	0.608	0.606	0.604	0.603	0.603	0.602	0.601	3/5
(1.25, 1.25)	0.857	0.852	0.847	0.844	0.842	0.840	0.838	5/6
(1.50, 0.15)	0.577	0.564	0.560	0.557	0.556	0.554	0.553	11/20
(1.50, 0.75)	0.772	0.764	0.760	0.758	0.756	0.754	0.753	3/4
(1.50, 1.50)	1.045	1.032	1.024	1.019	1.015	1.011	1.008	1

Table 2 Evaluation of the sequence (5). Since the values appear to converge as K grows, we conjecture that the $o_K(1)$ error term is $O(1/K)$

(c_a^2, c_s^2)	$K=50$	$K=100$	$K=150$	$K=200$	$K=250$	$K=300$
(0.4, 0.7)	−0.015098	−0.0150129	−0.0149844	−0.0149702	−0.0149616	−0.0149559
(0.4, 1.5)	0.315844	0.324518	0.32743	0.32889	0.329767	0.330352
(1.0, 1.0)	−0.326797	−0.330033	−0.331126	−0.331675	−0.332005	−0.332226
(1.3, 1.5)	0.658669	0.700843	0.715234	0.722493	0.726868	0.729794

state space of the PH/PH/1/K queue involves $7+(7+2)K$ states (over 900 states for $K=100$). The MAP specification labels certain transitions (service completions) and allows evaluating υ^* and λ^* for the corresponding point process. The major part of the computation is inversion of a matrix of a dimension equaling the number of states. Examples with $K=300$ can still be solved in fair time, but as K grows or c^2 approaches 0, the computation time grows cubically with K or c^{-2}.

Conjecture 1 also states that the $o_K(1)$ error term for finite K is $O(1/K)$. Note that this is in agreement with the explicit M/M/1/K result (2). We can further verify this for PH/PH/1/K queues by observing that the sequence,

$$K\left(\gamma^* - \frac{1}{3}\left(c_a^2+c_s^2\right)\right), \quad K=1,2,\ldots, \tag{5}$$

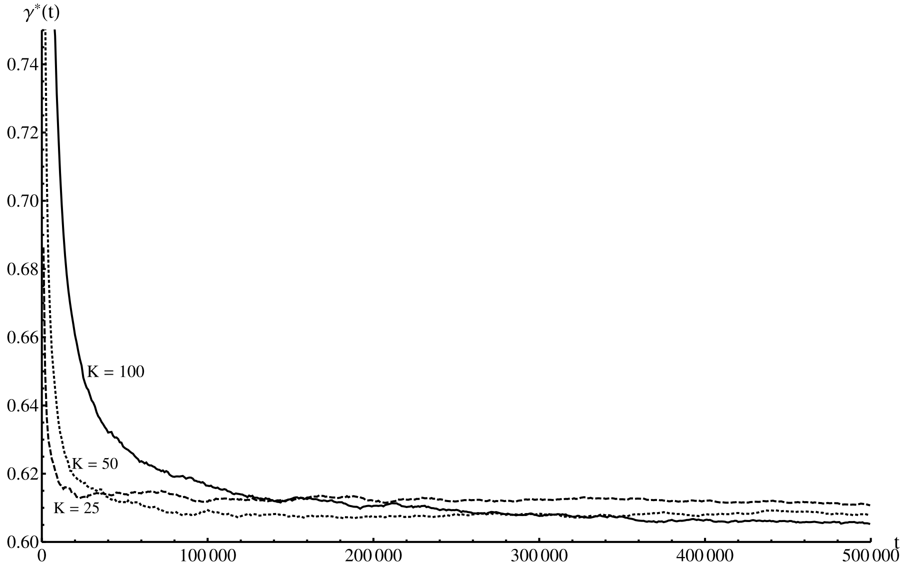

Fig. 2 Simulation estimates of $\gamma^*(t)$ for PH/PH/1/K queues with $c_a^2 = 0.6$ and $c_s^2 = 1.2$ for $K = 25, 50, 100$, $\mu = \lambda = 1$

converges to a constant. We do so for a few examples. The results are displayed in Table 2. We believe that the values in the right most column estimate B, where

$$\gamma^* = \frac{1}{3}\left(c_a^2 + c_s^2\right) + \frac{B}{K} + o\left(\frac{1}{K}\right).$$

Indeed, for the M/M/1/K case, the B that appears from the matrix-analytic computation agrees with $B = -1/3$ which appears in (2).

4 Simulation runs

In addition to the PH/PH/1/K numerics, we also performed simulations. For this we used a simple discrete event simulation of the queue, generating many independent realizations of $D(t)$. We use $\lambda = \mu = 1$ in all simulations and run each realization for a long time horizon: $T = 0.5 \times 10^6$, sampling the mean and the variance every 10^3 time units.[4] Observe first Fig. 2 where we plot estimates of $\gamma^*(t)$ for PH/PH/1/K queues with $c_a^2 = 0.6$ and $c_s^2 = 1.2$ for $K = 25, 50, 100$. These curves were obtained using approximately 0.4×10^6 runs for each K. It should be noted that we can essentially compute $\gamma^*(t)$ for PH/PH/1/K numerically (for every t). This involves matrix-exponential computations (cf. formula (10) of [13]).

[4] In an attempt to obtain the curves relating to a steady-state system, we 'warm up' the queue, for a duration of 0.2×10^6 before beginning the run of duration T, our warm up begins with $\lfloor K/2 \rfloor$ customers in the queue.

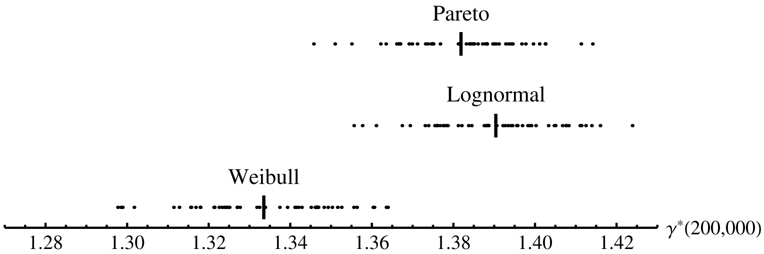

Fig. 3 The distribution of the estimators of $\gamma^*(0.2 \times 10^6)$. 50 estimators, each using 20,000 realizations, were obtained for each of the 3 service time distributions. The *vertical lines* signify the location of the means

Our conjecture for $K \to \infty$ states that $\gamma^* = \gamma^*(\infty) = 0.6$, we indeed see that the curves approach this for increasing K. Further note that it appears that for larger K the convergence of $\gamma^*(t)$ to γ^* is slower. This was also observed for the M/M/1/K queue in [13] (see Fig. 6 of that paper).

Our real interest in performing the simulations is to analyze distributions that cannot be represented as phase-type (with a finite number of phases). For this we tested a limited number of cases with $\lambda = \mu = 1$ and $c_a^2 = c_s^2 = 2$:

- *Bimodal distribution*, having a mass at $1/2$ w.p. $8/9$ and a mass at 5 w.p. $1/9$.
- *Heavy-tailed Weibull distribution.* $P(X > x) = e^{-(\frac{x}{\beta})^\alpha}$ with $\beta = \Gamma(1 + \frac{1}{\alpha})^{-1}$ and α being the positive solution of $\Gamma(1 + \frac{2}{\alpha}) = 3\Gamma(1 + \frac{1}{\alpha})^2$ where $\Gamma(\cdot)$ is the gamma function. This implies that $\alpha \approx 0.7209$ and $\beta \approx 0.81179$.
- *Lognormal distribution* with paraments $m = \log(3^{-1/2})$ and $\sigma^2 = \log(3)$, i.e., $X = e^Y$, where Y is normally distributed with mean m and variance σ^2.
- *Pareto(3) distribution* with support $[0, \infty)$. $P(X > x) = (1 + x/3)^{-4}$.

The results are presented in Fig. 1. Each curve was obtained by running 0.5×10^5 realizations. The main point of this figure was already discussed in the beginning of Sect. 2:

Open Problem *Heavy-tailed Weibull and light-tailed distributions exhibit one type of behavior, in agreement with Conjecture 1. Lognormal and Pareto exhibit a different behavior.*

To be sure that the difference between the two groups of curves is not a matter of statistical error, we also plot the distributions of the estimators of $\gamma^*(t)$. For this, we ran 50 repetitions of 20,000 runs, each for a duration of 0.2×10^6 time units. Each repetition resulted in a sample variance and a sample mean of the random variable $D(0.2 \times 10^6)$. The resulting estimators of $\gamma^*(t)$ are plotted in Fig. 3. As is implied by the figure, there is indeed a clear difference between the Weibull and the other two heavy-tailed cases, similarly to Fig. 1.[5]

[5]Note that Fig. 1 used 25 times more samples. Had we used this amount of samples for the runs of Fig. 3, the distributions of the estimators would have been even tighter.

5 Conclusion

We have presented a conjecture regarding GI/G/1/K queues along with some numerical evidence. Proving our conjecture will complete the picture of the variability parameter and the BRAVO effect for the basic single server queues. Proper (minimal) conditions on the inter-arrival and service time distributions are yet to be determined.

In general, the BRAVO phenomena is quite puzzling and somewhat counter intuitive.

Open Problem *A complete intuitive explanation is still lacking, and we pose this as a general open question.*

We point out that the factor $2/3$ arises in the asymptotic variance parameter of the local time at a barrier of driftless reflected Brownian motion with two barriers (see [4] and [19]). With the proper scaling, this should constitute the asymptotic variance of the idle time process which also equals the asymptotic variance of the busy time process. Further, we may possibly be able to make use of the following equality (cf. the proof of Theorem 3 in [14]):

$$\hat{D}^n(t) = \hat{S}^n\big(\bar{T}^n(t)\big) + \mu \hat{T}^n(t). \tag{6}$$

Here $S(t)$ is the renewal service process, $T(t)$ is the busy time, and the bar (fluid scaling) and hat (diffusion scaling) notation has the following meaning for a process $Z(t)$:

$$\bar{Z}^n(t) = \frac{Z(nt)}{n}, \quad \bar{Z}(t) = \lim_{t\to\infty} \bar{Z}^n(t), \quad \hat{Z}^n(t) = \frac{Z(nt) - \bar{Z}(nt)}{\sqrt{n}}.$$

It is possible that (6) along with the results of [4, 19] (see also [18]) will yield the proper insight and possibly lead the way to a proof of our conjecture.

Following the diffusion approximation framework of above, perhaps a more general formulation of Conjecture 1 can be formulated in terms of assumptions on the existence functional central limit theorems for the arrival and service processes. In fact, we believe that the BRAVO phenomena does not heavily depend on the i.i.d. assumptions of the inter-arrival and service sequences. In the other direction, an alternative more analytic approach, may be to use regenerative arguments to evaluate the asymptotic variance rate of say M/PH/1/K, M/G/1/K, PH/M/1/K or GI/M/1/K queues. Such an analysis can perhaps follow the renewal reward approach used in [14, Sect. 6.5] along with explicit results for joint distribution of the busy period and number of customers served in such queues (cf. [1] and references therein).

Acknowledgements I thank Ahmad Al-Hanbali, Daryl Daley, Onno Boxma, Andreas Löpker, Ward Whitt and Bert Zwart for useful comments and discussion. The bulk of this research was conducted while the author was a post-doctoral researcher at EURANDOM, Eindhoven University of Technology, partly supported by NWO-VIDI grant 639.072.072.

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