The BRAVO Effect in Queues

(and more stuff about variance of output counts)

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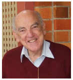
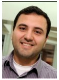
Variance Collaborators



Daryl Daley

Johan van Leeuwaarden



Ahmad Al-Hanbali

Michel Mandjes

Ward Whitt

Sophie Hautphenne

Yoav Kerner

Peter Taylor

Werner Scheinhardt

Brendan Patch

Thomas Taimre

Gideon Weiss

Queueing Models

$$M/M/1, \qquad M/M/1/K, \qquad M/M/s/K, \qquad M/M/s/K{+}M,$$

$$GI/G/1, \qquad GI/G/1/K, \quad \ldots$$

Basic conservation equation for a single queue

$$Q(t) = Q(0) + \Big(A(t) - L(t)\Big) - \Big(R(t) + D(t)\Big)$$

The Output Process $D(\cdot)$

$$D(t) = Q(0) + \Big(A(t) - L(t) - R(t)\Big) - Q(t)$$

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- Production
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Some performance measures of interest

- The law of $\{D(t),\ t \geq 0\}$
- $\mathbb{E}[D(t)]$, $\mathrm{Var}\big(D(t)\big)$
- $\lambda^* := \lim_{t\to\infty} \frac{\mathbb{E}[D(t)]}{t}$, $\quad \overline{V} := \lim_{t\to\infty} \frac{\mathrm{Var}\big(D(t)\big)}{t}$, $\quad \mathcal{D} := \frac{\overline{V}}{\lambda^*}$
- Asymptotic normality: $D(t) \sim \mathcal{N}\Big(\lambda^* t,\ \overline{V} t\Big)$, large t
- Second order approximations, e.g.,
$$\mathrm{Var}\big(D(t)\big) = \overline{V} t + \overline{b} + o(1)$$
- Asymptotic covariances, etc...

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- What is $\mathcal{D} = \lim_{t\to\infty} \frac{\mathrm{Var}\big(D(t)\big)}{\mathbb{E}[D(t)]}$ for queues?

- E.g. in stationary (and thus stable M/M/1): $\mathcal{D} = 1$

For illustration, consider M/M/1/K

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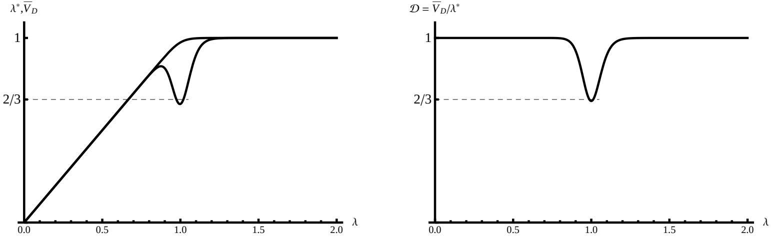

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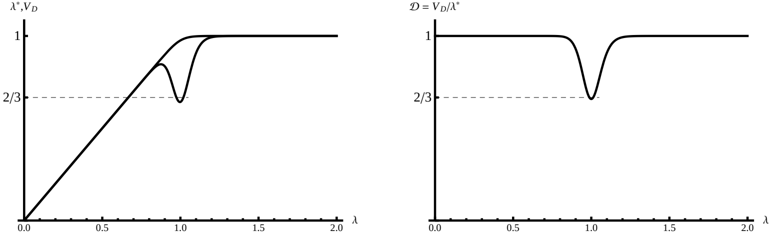


We call this **BRAVO**:

Balancing **R**educes **A**symptotic **V**ariance of **O**utputs

Finite Birth-Death Asymptotic Variance (and BRAVO)

Finite Birth-Death Setting

- Irreducible birth-death process on finite state space
- Birth rates: $\lambda_0, \dots, \lambda_{J-1}$
- Death rates: $\mu_1, \dots, \mu_J$
- Stationary distribution: $\pi_0, \dots, \pi_J$
- $D(t)$ is number of downward transitions (deaths) during $[0, t]$, each "filtered" independently with state-dependent probabilities, $q_1, \dots, q_J$.
- e.g. The output process (served customers) in M/M/s/K+M :

$$\lambda_i = \lambda, \quad \mu_i = \mu\,(i \wedge s) + \gamma\,(i - s)^+, \qquad q_i = \frac{\mu\,(i \wedge s)}{\mu\,(i \wedge s) + \gamma\,(i - s)^+}, \quad i = 0, 1, \dots, s+K$$

Of interest:

$$\mathcal{D} = \frac{\overline{V}}{\lambda^*} = \lim_{t \to \infty} \frac{\mathrm{Var}\big(D(t)\big)}{\mathbb{E}[D(t)]}$$

Finite Birth-Death Asymptotic Variance Formula

Theorem: Daryl Daley, Johan van Leeuwaarden, Y.N. 2014

$$\mathcal{D} := \lim_{t\to\infty} \frac{\mathrm{Var}\big(D(t)\big)}{\mathbb{E}[D(t)]} = 1 - 2\sum_{i=0}^{J}(P_i - \Lambda_i^*)\Big(q_{i+1} - \frac{\lambda^*}{\pi_i\lambda_i}(P_i - \Lambda_i^*)\Big),$$

with,

$$P_i := \sum_{j=0}^{i} \pi_j, \qquad \lambda^* := \sum_{j=1}^{J} \mu_j q_j \pi_j, \qquad \Lambda_i^* := \frac{\sum_{j=1}^{i} \mu_j q_j \pi_j}{\lambda^*}.$$

Note: In Y.N. and Weiss 2008, similar expression for case $q_i \equiv 1$

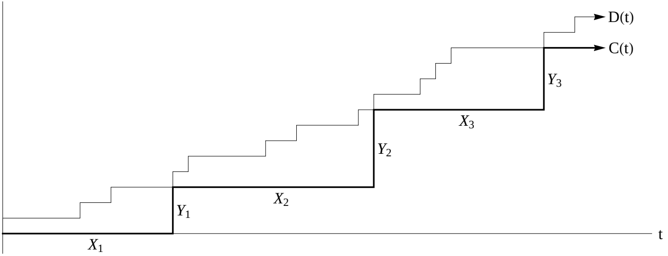
Note: In case $\lambda_i \equiv \lambda$, $q_i \equiv 1$:

$$\mathcal{D} = 1 - 2\frac{\pi_J}{1-\pi_J}\sum_{i=0}^{J} P_i\Big(1 - \pi_J\frac{P_i}{\pi_i}\Big)$$

Idea of Renewal Reward Derivation

"Embed" $D(t)$ in a Renewal-Reward Process, $C(t)$

1. $(X_n, Y_n) \equiv$ (busy cycle, number served) in cycle n
2. $N(t) = \sup\{n : \sum_{i=1}^{n} X_i \leq t\}$, $C(t) = \sum_{i=1}^{N(t)} Y_i$
3. Asymptotic variance rates of $C(t)$ and $D(t)$ are equal
4. Known:
   - Asymptotic variance rate of $C(t)$ is $\frac{1}{\mathbb{E}[X]}\mathrm{Var}\left(Y - \frac{\mathbb{E}[Y]}{\mathbb{E}[X]}X\right)$
   - Systems of equations for
     1'st, 2'nd and cross moments of X and Y



Back to the M/M/1/K Queue

Here π_i is truncated geometric distribution when $\lambda \neq \mu$ and a uniform distribution when $\lambda = \mu$

Using $\mathcal{D} = 1 - 2\frac{\pi_J}{1-\pi_J}\sum_{i=0}^{J} P_i\left(1 - \pi_J\frac{P_i}{\pi_i}\right)$:

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$$\mathcal{D} = \begin{cases} 1 + o_K(1), & \lambda \neq \mu, \\ \frac{2}{3} + o_K(1), & \lambda = \mu. \end{cases}$$

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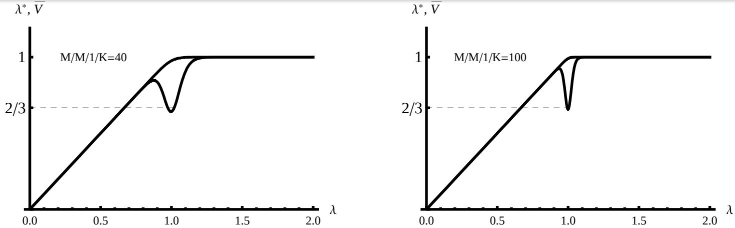

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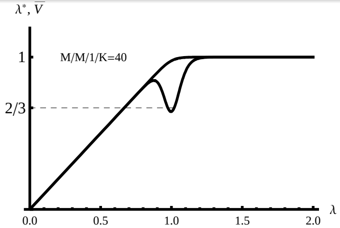


In fact, for any λ, μ, we have an explicit expression for $\mathcal{D}$ (alt. $\overline{V}$, λ^*) and even for $\overline{b}$ in,

$$\mathrm{Var}\big(D(t)\big) = \overline{V}t + \overline{b} + o(1)$$

Multi-Server Systems in the Halfin-Whitt (QED) Regime

Quality and Efficiency Driven (QED) Scaling Regime

A sequence of systems

Consider a sequence of $M/M/s/K$ queues with increasing $s = 1, 2, \ldots$ and with $\rho_s := \frac{\lambda}{s\mu}$ and K_s such that,

$$(1-\rho_s)\sqrt{s} \to \beta \in (-\infty, \infty)$$
$$\frac{K_s}{\sqrt{s}} \to \eta \in (0, \infty)$$

So for large s:

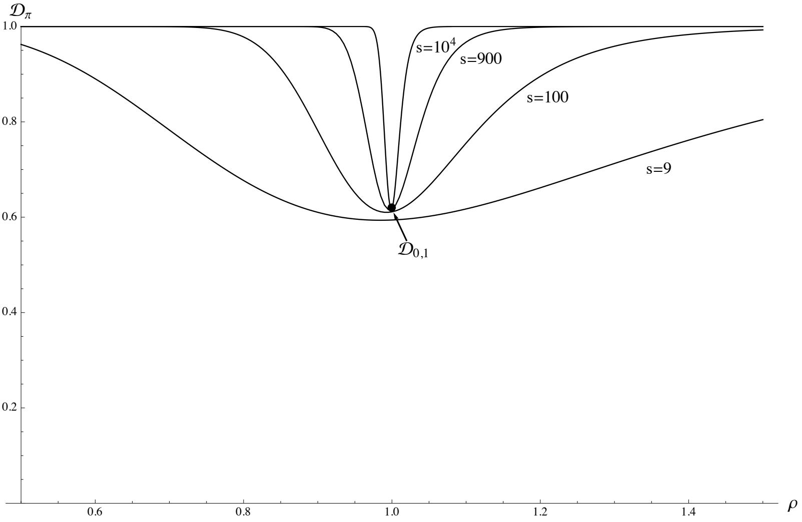
$$\rho_s \approx 1 - \beta/\sqrt{s}$$
$$K_s \approx \eta\sqrt{s}$$

Halfin, Whitt, 1981, Garnett, Mandelbaum, Reiman 2002, Borst, Mandelbaum, Reiman, 2004, Whitt, 2004, Pang, Talreja, Whitt, 2007, Janssen, van Leeuwaarden, Zwart, 2011, Kaspi, Ramanan 2011...

Favorable QED Properties

- Probability of delay converges to a value $\in (0, 1)$
- Mean waiting times are typically $O(s^{-1/2})$
- Large queue lengths almost never occur
- Quick mixing times
- In applications: Call-centers (etc...) describes behavior well and allows for asymptotic approximate optimization of staffing etc...
- How about BRAVO?

$M/M/s/\lfloor\sqrt{s}\rfloor$



M/M/s/K QED BRAVO

Theorem: Daryl Daley, Johan van Leeuwaarden, Y.N. 2013

Consider QED scaling with $\beta \neq 0$:

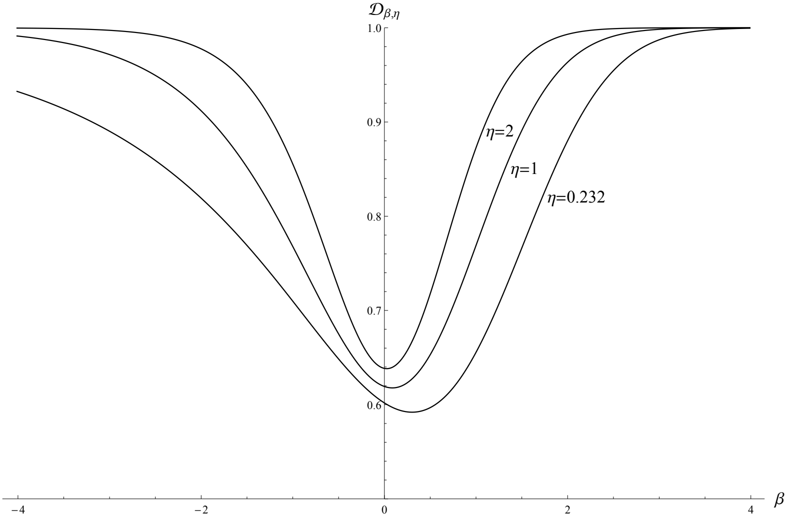
$$\mathcal{D}_{\beta,\eta} := \lim_{s,K\to\infty} \lim_{t\to\infty} \frac{Var\big(D(t)\big)}{\mathbb{E}\big(D(t)\big)},$$

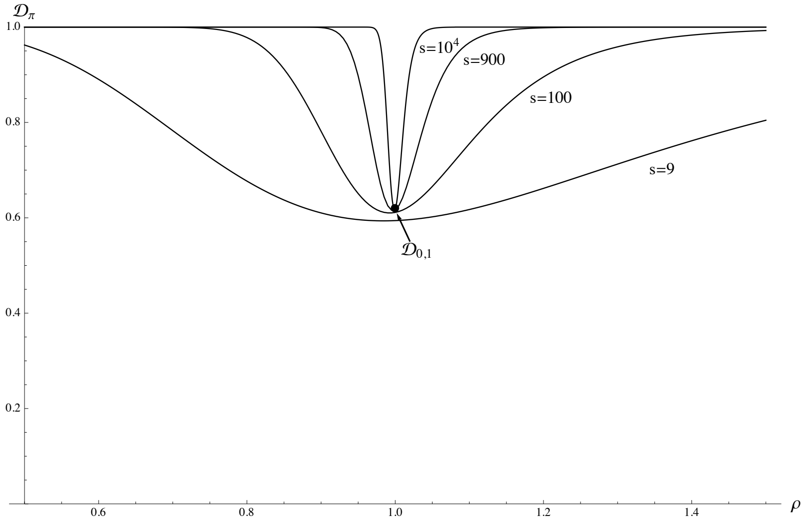
$$\mathcal{D}_{\beta,\eta} = 1 - \frac{2\beta^2 e^{-\beta\eta} h^2}{\phi(\beta)} \int_{-\beta}^{\infty} \left(1 - \beta e^{-\beta\eta} h \frac{\Phi(-u)}{\phi(u)}\right) \Phi(-u)\, du$$
$$+ 2e^{-\beta\eta} h(1 + e^{-\beta\eta} h)\Big(1 - \beta\eta - e^{-\beta\eta} + (1 - 2\beta\eta e^{-\beta\eta} - e^{-2\beta\eta})h\Big)$$

where

$$h = \lim_{s\to\infty} \frac{\mathbb{P}\big(Q_s \geq s\big)}{1 - e^{-\beta\eta}} = \frac{1}{1 - e^{-\beta\eta} + \frac{\beta\Phi(\beta)}{\phi(\beta)}}$$

BRAVO Viewed Through the QED Lens





$\mathcal{D}_\pi$
1.0
0.8
0.6
0.4
0.2
s=10^4
s=900
s=100
s=9
$\mathcal{D}_{0,1}$
0.6
0.8
1.0
1.2
1.4
ρ

M/M/s/K QED BRAVO with $\rho \equiv 1$ $(\beta = 0)$

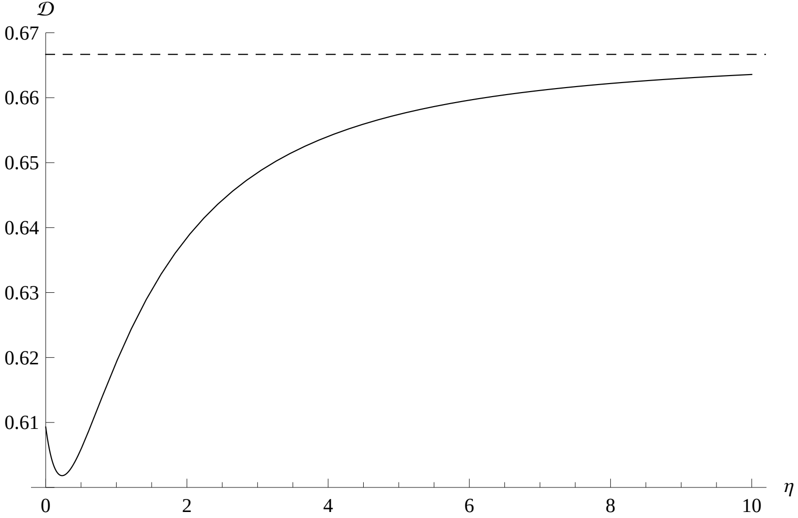
Theorem: Daryl Daley, Johan van Leeuwaarden, Y.N. 2013

Assume $\rho \equiv 1$ and $\frac{K_s}{\sqrt{s}} \to \eta \in (0, \infty)$. Then

$$\mathcal{D}_{0,\eta} := \lim_{s,K\to\infty} \lim_{t\to\infty} \frac{Var\big(D(t)\big)}{\mathbb{E}\big(D(t)\big)},$$

$$\mathcal{D}_{0,\eta} = \frac{2}{3} - \frac{\left(6 - \frac{3\pi}{2}\right)\eta - \frac{1}{2}\pi\sqrt{\frac{\pi}{2}} + 3\sqrt{2\pi}(1 - \log 2)}{3\left(\eta + \sqrt{\frac{\pi}{2}}\right)^3}.$$

M/M/s/$\lfloor \eta\sqrt{s} \rfloor$ $\quad s \to \infty \quad$ at $\rho \equiv 1$ ($\beta = 0$)



Idea of BRAVO QED Derivations

Use

$$\mathcal{D} = 1 - 2\frac{\pi_J}{1-\pi_J}\sum_{i=0}^{J} P_i\left(1 - \pi_J\frac{P_i}{\pi_i}\right).$$

Using QED scaling:

$$(1-\rho_s)\sqrt{s} \to \beta, \qquad \frac{K_s}{\sqrt{s}} \to \eta,$$

evaluate the limit,

$$\lim_{s,K\to\infty} \frac{\pi_J^{(s,K)}}{1-\pi_J^{(s,K)}}\sum_{i=0}^{J} P_i^{(s,K)}\left(1 - \pi_J^{(s,K)}\frac{P_i^{(s,K)}}{\pi_i^{(s,K)}}\right).$$

Beyond Finite Birth-Death Queues

M/M/1 Queue

When $K = \infty$, the birth-death $\mathcal{D}$ formula, generally does not hold. In this case,

$$\mathcal{D} = \begin{cases} 1, & \lambda \neq \mu, \\ ?, & \lambda = \mu. \end{cases}$$

A guess is $\frac{2}{3}$, since for $K < \infty$, $\mathcal{D} = \frac{2}{3} + o_K(1)$ . . .

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Theorem:
Ahmad Al-Hanbali, Michel Mandjes, Y. N., Ward Whitt, 2011

For the M/M/1 queue with $\lambda = \mu$ and arbitrary initial conditions of $Q(0)$ (with finite second moments),

$$\mathcal{D} = 2\Big(1 - \frac{2}{\pi}\Big) \approx 0.727.$$

Proof based on analysis of classic Laplace transform of the generating function of $D(\cdot)$

M/M/1 Queue

$\mathrm{Var}\big(D(t)\big) =$ horrible expression involving integrals of Bessel functions

From it:

$$\mathrm{Var}\big(D(t)\big) = \begin{cases} \lambda t - \frac{\rho}{(1-\rho)^2} + o(1), & \text{if } \ \lambda < \mu, \\ 2(1 - \frac{2}{\pi})\lambda t - \sqrt{\frac{\lambda}{\pi}}\, t^{1/2} + \frac{\pi - 2}{4\pi} + o(1), & \text{if } \ \lambda = \mu, \\ \mu t - \frac{\rho}{(1-\rho)^2} + o(1), & \text{if } \ \lambda > \mu, \end{cases}$$

The Stable M/G/1 Queue

Theorem: Sophie Hautphenne, Yoav Kerner, Y. N., Peter Taylor, 2013

Consider the stable M/G/1 queue with finite third service moment, parameterized by (*arrival rate, load, scv, skewness*)$= (\lambda, \rho, c^2, \gamma)$.

Stationary version:

$$\mathrm{Var}\big(D(t)\big) = \lambda t + L_e \frac{\rho}{(1-\rho)^2} + o(1),$$

$$L_e = \frac{(3c^4 - 4\gamma c^3 + 6c^2 - 1)\rho^3 + (4\gamma c^3 - 12c^2 + 4)\rho^2 + (6c^2 - 6)\rho}{6}.$$

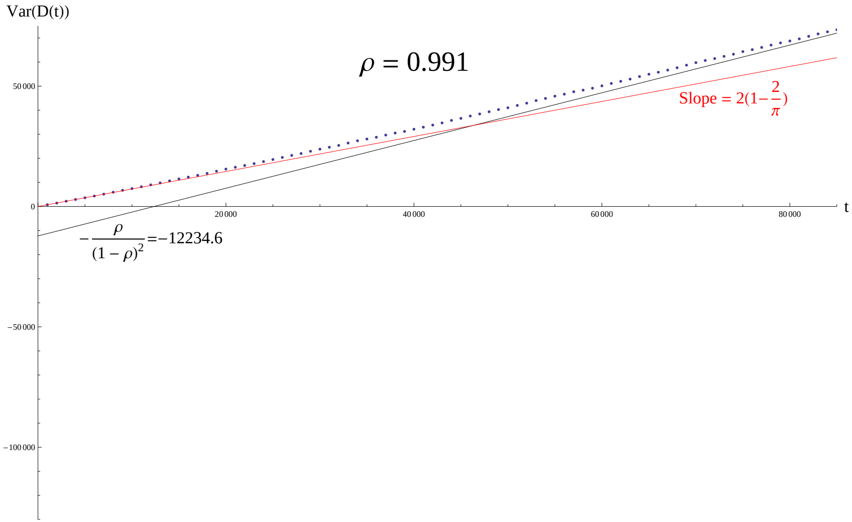
Starting empty version:

$$\mathrm{Var}\big(D(t)\big) = \lambda t - (1 - L_0) \frac{\rho}{(1-\rho)^2} + o(1),$$

$$L_0 = \frac{(3c^4 - 4\gamma c^3 + 6c^2 - 1)\rho^3 + (4\gamma c^3 - 6c^2 - 2)\rho^2 - (6c^2 - 6)\rho}{12}.$$

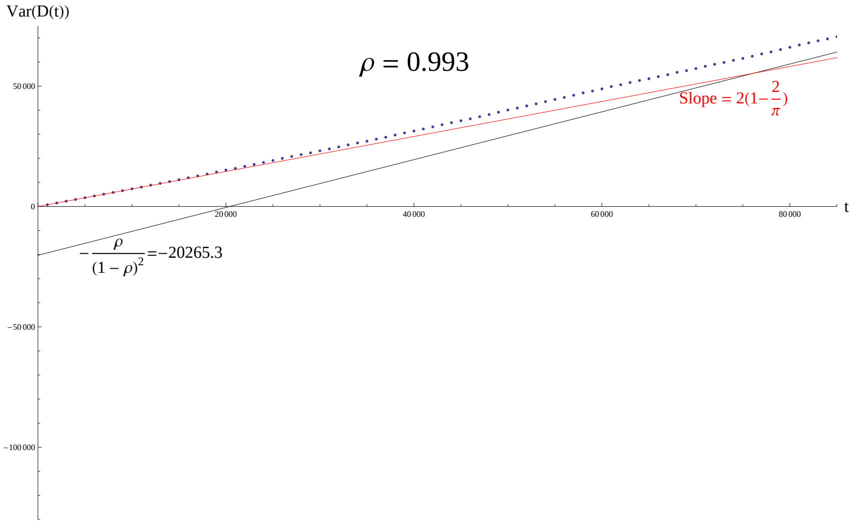
M/M/1: $c^2 = 1, \gamma = 2$. $L_e = 0$, $L_0 = 0$.

M/M/1 Queue



Var(D(t))
$\rho = 0.991$
50 000
0
−50 000
−100 000
20 000
40 000
60 000
80 000
t
Slope = $2(1-\frac{2}{\pi})$
$-\frac{\rho}{(1-\rho)^2}$=−12234.6

M/M/1 Queue



Var(D(t))
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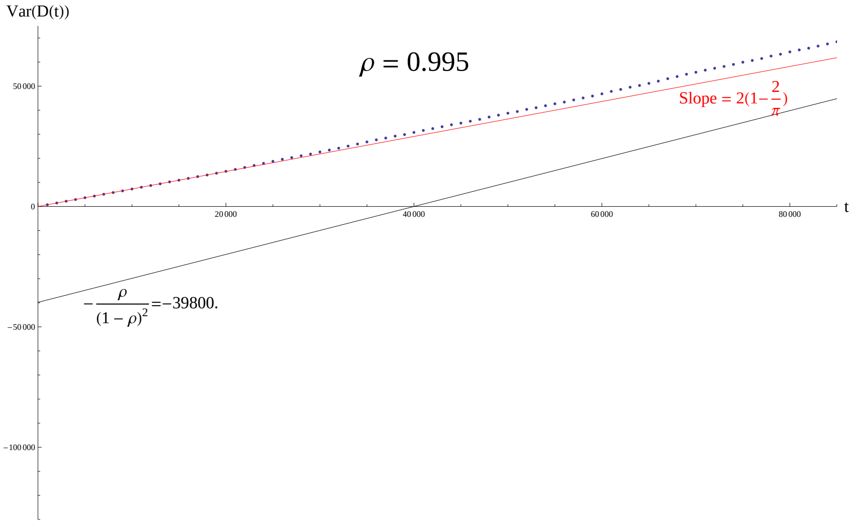
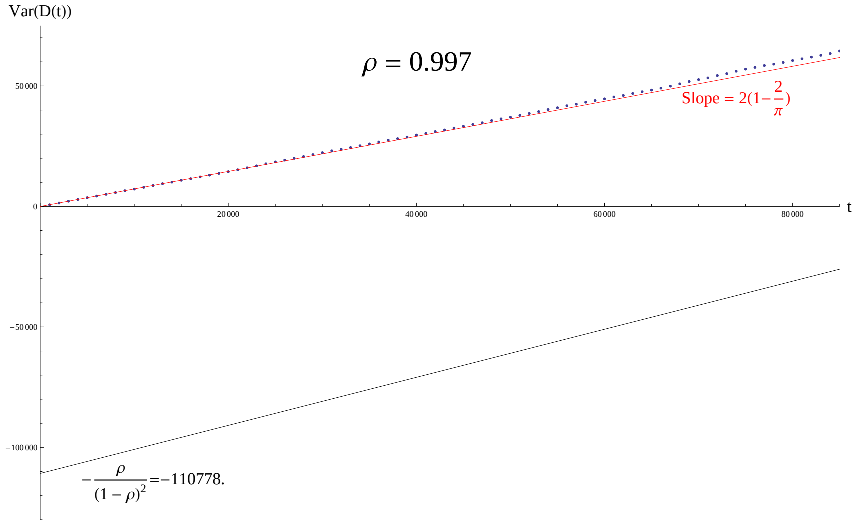
50 000
0
−50 000
−100 000
20 000
40 000
60 000
80 000
t
Slope = $2(1-\frac{2}{\pi})$
$-\frac{\rho}{(1-\rho)^2}=-20265.3$

M/M/1 Queue


Var(D(t))
$\rho = 0.995$
Slope = $2(1-\frac{2}{\pi})$
$-\frac{\rho}{(1-\rho)^2}$=−39800.
t
50 000
0
−50 000
−100 000
20 000
40 000
60 000
80 000

M/M/1 Queue



Var(D(t))
$\rho = 0.997$
50 000
0
−50 000
−100 000
20 000
40 000
60 000
80 000
t
Slope = $2(1-\frac{2}{\pi})$
$-\frac{\rho}{(1-\rho)^2}$=−110778.

GI/G/1 Queue

Moving away from the memory-less assumptions,

$$\mathcal{D} = \begin{cases} c_a^2, & \lambda < \mu, \\ ?, & \lambda = \mu, \\ c_s^2, & \lambda > \mu. \end{cases}$$

For M/M/1 it was $2(1 - \frac{2}{\pi})$...

GI/G/1 Queue

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For M/M/1 it was $2(1 - \frac{2}{\pi})$...

Theorem:
Ahmad Al-Hanbali, Michel Mandjes, Y.N., Ward Whitt, 2011

For the GI/G/1 queue with $\lambda = \mu$, arbitrary finite second moment initial conditions $\big(Q(0), V(0), U(0)\big)$, finite fourth moments of the inter-arrival and service times, and $\mathbb{P}(B > x) \sim L(x)x^{-1/2}$, where B denotes the busy period and $L(\cdot)$ is a slowly varying function,

$$\mathcal{D} = (c_a^2 + c_s^2)\Big(1 - \frac{2}{\pi}\Big).$$

Proof using diffusion limit of $(D(n\cdot) - \lambda n\cdot)/\sqrt{\lambda n\cdot}$ as $n \to \infty$ (Iglehart and Whitt 1971).

GI/G/1/K Queue

$$\mathcal{D} = \begin{cases} c_a^2 + o_K(1), & \lambda < \mu, \\ ?, & \lambda = \mu, \\ c_s^2 + o_K(1), & \lambda > \mu. \end{cases}$$

For $M/M/1/K$ it was $\frac{2}{3} + o_K(1)$, for $GI/G/1$ it was $(c_a^2 + c_s^2)(1 - \frac{2}{\pi})$...

GI/G/1/K Queue

$$\mathcal{D} = \begin{cases} c_a^2 + o_K(1), & \lambda < \mu, \\ ?, & \lambda = \mu, \\ c_s^2 + o_K(1), & \lambda > \mu. \end{cases}$$

For $M/M/1/K$ it was $\frac{2}{3} + o_K(1)$, for $GI/G/1$ it was $(c_a^2 + c_s^2)(1 - \frac{2}{\pi})$...

Conjecture (numerically tested): Y.N., 2011

For the GI/G/1/K queue with $\lambda = \mu$ and arbitrary initial conditions and light-tailed service and inter-arrival times,

$$\mathcal{D} = (c_a^2 + c_s^2)\frac{1}{3} + O\left(\frac{1}{K}\right).$$

Numerical verification done by representing the system as PH/PH/1/K MAPs

Wrap Up

Summary

Known BRAVO constants:

- Single server finite buffer: $2/3$
  (for GI/G replace 2 by $c_a^2 + c_s^2$)
- Single server infinite buffer $2(1 - 2/\pi)$:
  (for GI/G replace 2 by $c_a^2 + c_s^2$)
- Memoryless many servers finite buffer: $\mathcal{D}_{0,\eta} \in [0.6, 2/3]$

Not yet known:

- Formulas for asymptotic variance when $\rho \neq 1$ in other models
- Memoryless many servers infinite buffer (M/M/s)
- Many servers without memoryless assumptions (GI/G/s)
- Systems with reneging or other packet loss mechanisms
  (e.g. M/M/s/K+M in QED – work in progress)

Other questions: How can BRAVO be harnessed in practice?
Why does BRAVO occur?

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