The Asymptotic Variance Rate of the Output Process of Finite Capacity Birth-Death Queues

Yoni Nazarathy, Gideon Weiss
Department of Statistics, The University of Haifa

Setting

Single station queue with finite capacity, represented by a birth-death Markov jump process:

- Birth rates: $\lambda_0, \ldots, \lambda_{K-1}$
- Death rates: $\mu_1, \ldots, \mu_K$
- Steady state distribution: $\boldsymbol{\pi} = (\pi_0, \ldots, \pi_K)$
- Assume queue is stationary
- Output process, $D(t) =$ number of jobs that have received service up to time t
- For $K > 1$, $D(t)$ is not a renewal process
- $\mathbb{E}\,[D(t)] = \lambda^* t, \quad \lambda^* = \sum_{i=1}^{K} \pi_i \mu_i$

Variance grows asymptotically linearly:

$$\text{Var}(D(t)) = \overline{V}\, t + o(t).$$

$\overline{V}$ is the asymptotic variance rate.

Main Theorem

A formula for the asymptotic variance rate:

$$\overline{V} = \lambda^* + \sum_{i=0}^{K-1} v_i$$

$$v_i = 2(M_i + \frac{M_i^2}{\pi_i \lambda_i}), \qquad M_i = \sum_{j=1}^{i} \pi_j \mu_j - \lambda^* \sum_{j=0}^{i} \pi_j.$$

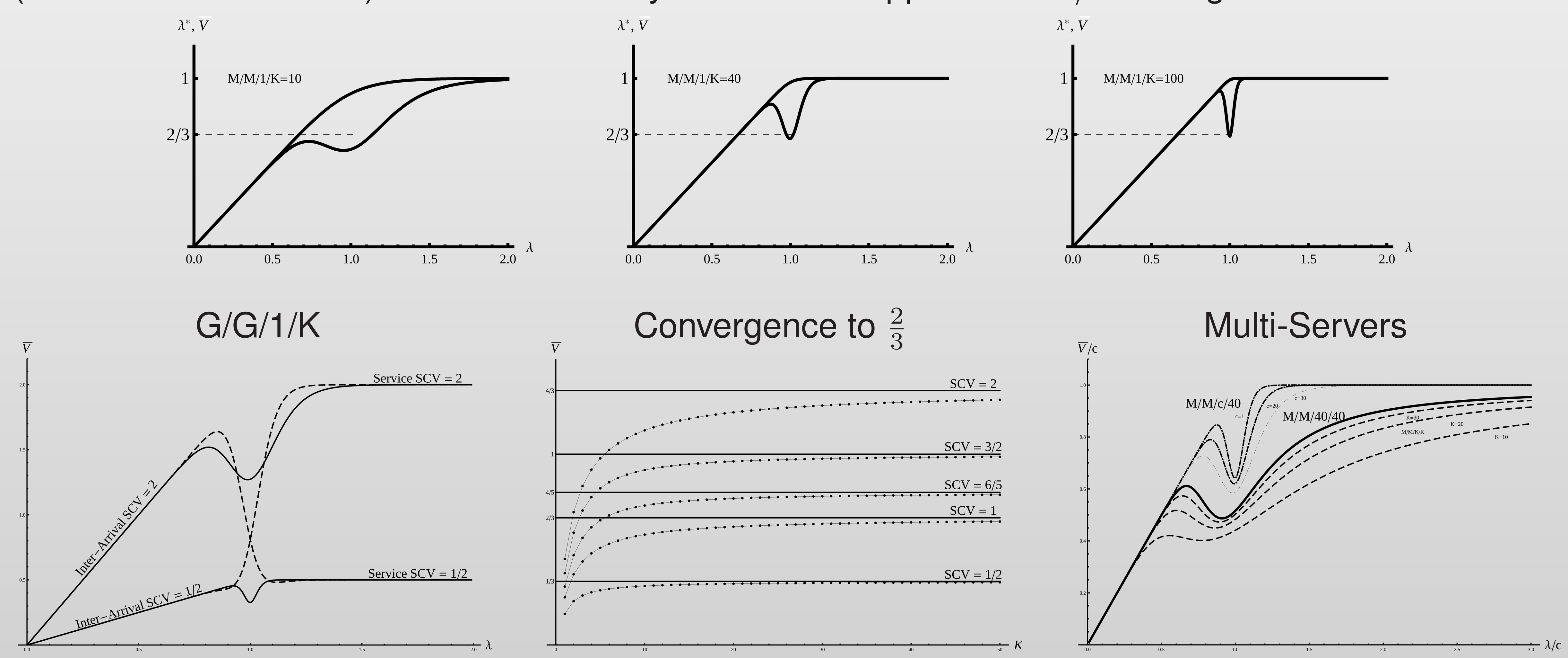
As a consequence:

- When $\lambda_0 \geq \ldots \geq \lambda_{K-1}$ and $\mu_1 \leq \ldots \leq \mu_K$, then $v_i < 0$, thus $\overline{V}/\lambda^* < 1$
- For $M/M/1/K$ with $\rho = \lambda/\mu$:

$$\overline{V} = \begin{cases} \lambda \frac{2K^2+K}{3K^2+6K+3} & \rho = 1 \\ \lambda \frac{(1+\rho^{K+1})(1-(1+2K)\rho^K(1-\rho)-\rho^{2K+1})}{(1-\rho^{K+1})^3} & \rho \neq 1 \end{cases}$$

BRAVO Effect – Balancing Reduces Asymptotic Variance of Outputs

$\overline{V}$ is locally minimized when the arrival rate equals the maximal service rate. In single server systems (M/M/1/K and GI/G/1/K) the reduction is by a factor that approaches $2/3$ for large K.



Numerical Computation of $\overline{V}$ using MAPs - Markov Arrival Processes

Generator matrix:

$$\Lambda = \underbrace{\begin{pmatrix} -\lambda_0 & \lambda_0 & & & 0 \\ \mu_1 & -(\mu_1+\lambda_1) & \lambda_1 & & \\ & \mu_2 & \ddots & \ddots & \\ & & \ddots & \ddots & \lambda_{K-1} \\ 0 & & & \mu_K & -\mu_K \end{pmatrix}}_{C} + \underbrace{\begin{pmatrix} 0 & 0 & & & 0 \\ \mu_1 & & & & \\ & \mu_2 & \ddots & \ddots & \\ & & \ddots & \ddots & 0 \\ 0 & & & \mu_K & 0 \end{pmatrix}}_{E}$$

$$\mathbb{E}\,[D(t)] = \underbrace{\boldsymbol{\pi} E \mathbf{1}}_{\lambda^*} t \qquad \text{Var}(D(t)) = \underbrace{\lambda^* - 2(\lambda^*)^2 - 2\boldsymbol{\pi} E \Lambda^- E \mathbf{1}}_{\overline{V}}\, t + \underbrace{2(\lambda^*)^2 - 2\boldsymbol{\pi} E \Lambda^- \Lambda^- E \mathbf{1}}_{\overline{B}} + O(t^{3r+2} e^{-bt})$$

No explicit formula for $\Lambda^- = (\Lambda - \mathbf{1}\boldsymbol{\pi})^{-1}$.

Main Proof Steps

- $M(t) =$ admissions $+$ outputs. $\overline{V}_M = 4\overline{V}$.
- $M(t)$ (also a MAP) has the same variance as an associated Markov Modulated Poisson Process (MMPP).
- Whitt – a closed formula for calculation of the asymptotic variance of birth-death MMPPs.

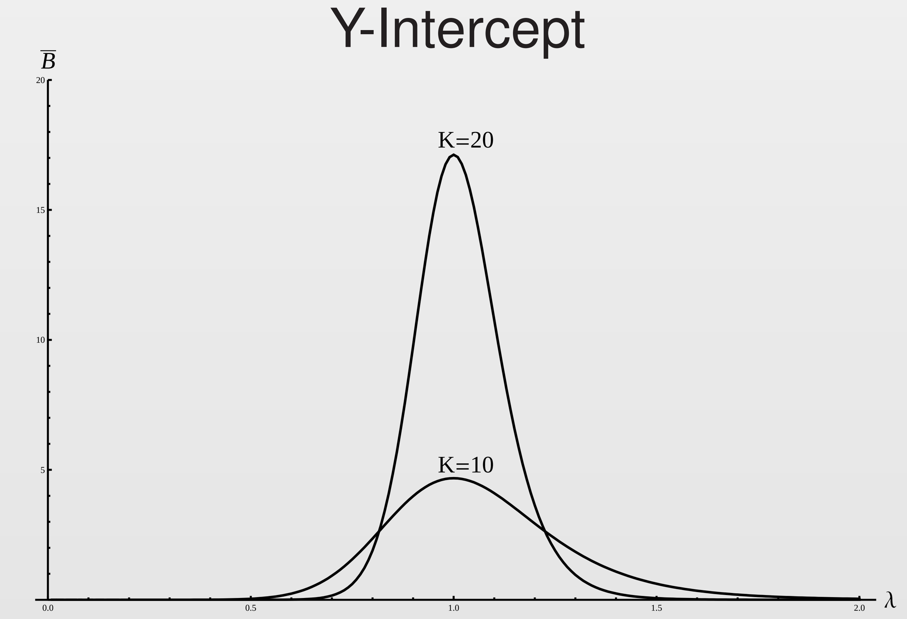
Key References

[1] S. Naryana and M. F. Neuts. The first two moment matrices of the counts for the Markovian arrival process. Stochastic Models, 1992.

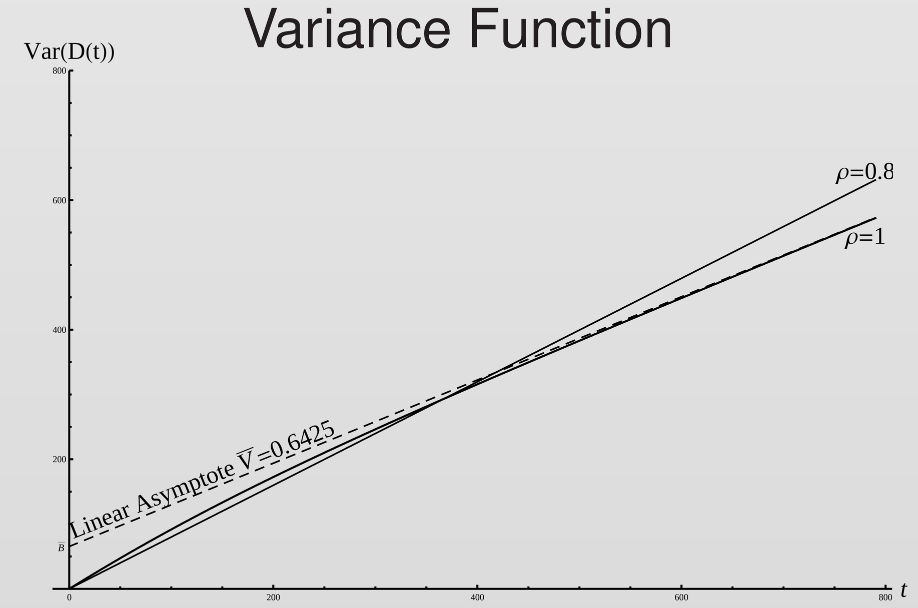
[2] W. Whitt. Asymptotic formulas for Markov processes with applications to simulation. Operations Research, 1992.

Variance for Finite t

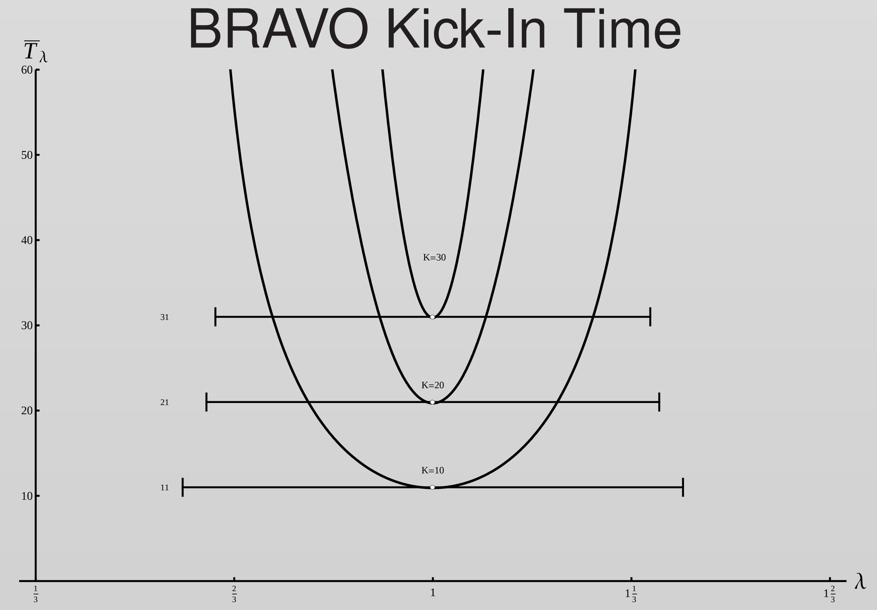
Y-Intercept



Variance Function



BRAVO Kick-In Time



Correlation of Outputs and Overflows