Introduction to Markov Chains

Today's Plan

In first 15 minutes

- Brief review
- The idea of a Markov Chain
- An introductory example
- Multi-step transitions
- Numerical illustration

In next 35 minutes *(will not be presented)*

- Further classical examples
- Overview of other properties of interest:
  - Class structure
  - Absorbtion probabilities
  - Mean hitting times
  - Stationary and limiting distributions
- Illustrations of complications that may arise with infinite state spaces

Brief Review

- Discrete time stochastic process: a sequence of random variables, $\{X_0, X_1, X_2, \ldots\}$ taking values in some *state space*, $\mathcal{S}$
- Simple example is a sequence i.i.d. random variables
- Gaussian random processes

Markov Chain Definition

- The process $\{X_0, X_1, X_2, \dots\}$ is a Markov chain if it adheres to the following:
  - Lack of memory:

$$\mathbb{P}\big(X_{n+1} = j | X_n = i_n, \dots, X_0 = i_0\big) = \mathbb{P}\big(X_{n+1} = j | X_n = i_n\big)$$

  - Time Homogeneity:

$$\mathbb{P}\big(X_{n+1} = j | X_n = i\big) = \mathbb{P}\big(X_1 = j | X_0 = i\big) := p_{i,j}$$

- When $|\mathcal{S}| = M < \infty$ the $M \times M$ matrix P with elements $p_{i,j}$ is called the transition probability matrix. This is a *stochastic matrix* (non-negative elements and rows sum up to 1)
- The parameters of the Markov chain are the stochastic matrix P and the distribution of X_0

A Simple Example: Weather Chain

- Weather modeling is a complex subject yet as a first approximation we may model the weather by a simple Markov chain
- Assume each time step represents one day
- Assume $\mathcal{S} = \{1 = \text{Rain},\ 2 = \text{Normal},\ 3 = \text{Sunny}\}$
- Assume that historical data shows:

$$P = \begin{bmatrix} 0.6 & 0.3 & 0.1 \\ 0.3 & 0.2 & 0.5 \\ 0.1 & 0.3 & 0.6 \end{bmatrix}$$

- Assume $\mathbb{P}\big(X_0 = 3\big) = 1$
- Lets do the following:
  - Draw a weighted directed graph associated with this Markov chain
  - Simulate a sample-path
- Question: As time evolves, where does this chain spend more time, in state 1 or state 3?

Multi-Step Transitions

$$
\begin{aligned}
p_{i,j}^{(2)} := \mathbb{P}\big(X_2 = j | X_0 = i\big) &= \sum_{k \in \mathcal{S}} \mathbb{P}\big(X_2 = j | X_1 = k, X_0 = i\big)\, \mathbb{P}\big(X_1 = k | X_0 = i\big) \\
&= \sum_{k \in \mathcal{S}} \mathbb{P}\big(X_2 = j | X_1 = k\big)\, \mathbb{P}\big(X_1 = k | X_0 = i\big) \\
&= \sum_{k \in \mathcal{S}} \mathbb{P}\big(X_1 = j | X_0 = k\big)\, \mathbb{P}\big(X_1 = k | X_0 = i\big) \\
&= \sum_{k \in \mathcal{S}} p_{k,j}\, p_{i,k}
\end{aligned}
$$

$$
p_{i,j}^{(2)} = \sum_{k=1}^{M} p_{i,k}\, p_{k,j}
$$

Denote $P^{(2)}$ the matrix of $p_{i,j}^{(2)}$. Thus: $P^{(2)} = P \cdot P = P^2$

Multi step transitions probabilities are obtained by matrix powers!

$$
P^{(n)} = P^n
$$