New Stability and Diffusion Results for Multi-Class Queueing Networks

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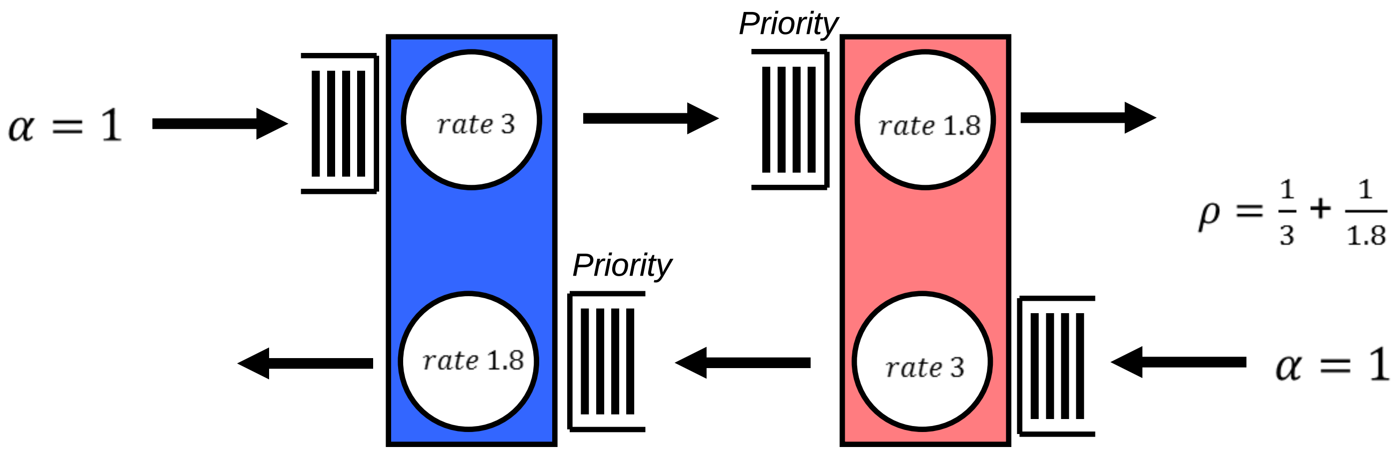
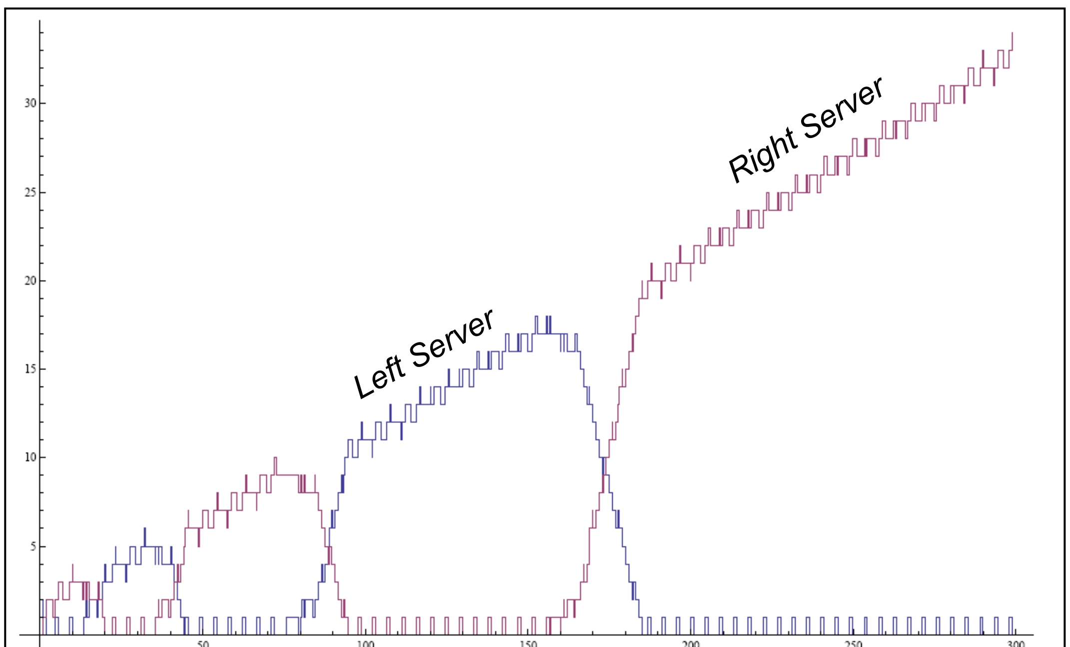
INTRODUCTION: STABILITY OF MULTI-CLASS QUEUEING NETWORKS

The $K_{umar}S_{eidman}R_{ybko}S_{toylar}$ Queueing Network (90's)



$$\rho = \frac{1}{3} + \frac{1}{1.8} = 0.88 < 1$$



In this talk

- Outline past research on Multi-Class Queueing Networks with Infinite Supplies

  - *N., Weiss, 2008-2009*

- Overview of New Results

  - Stability of certain examples

  - Diffusion Limits

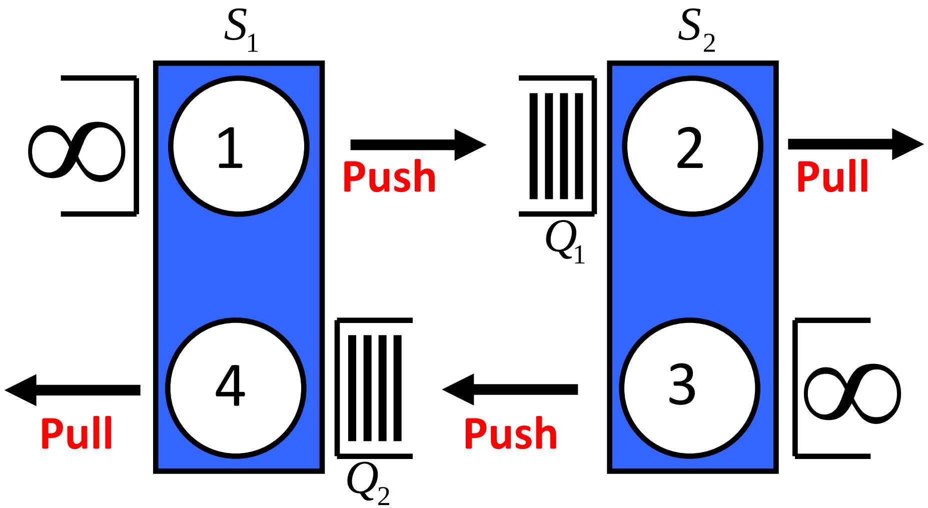
- Outlook

  - Stability of Queueing Networks is "hard"

  - Need methods to construct general policies

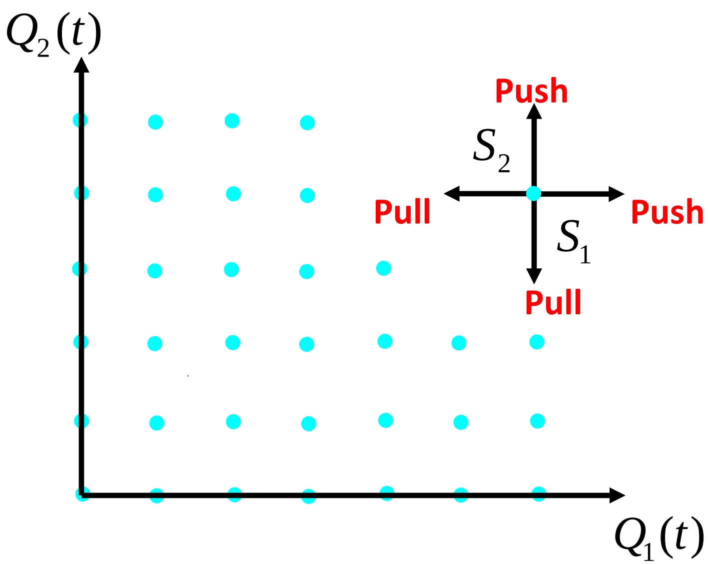
MULTI-CLASS QUEUEING NETWORKS WITH INFINITE SUPPLIES

Example: The Push-Pull Network

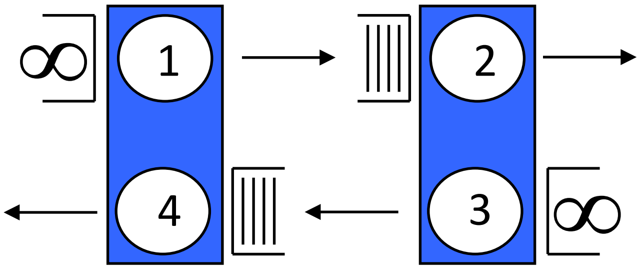


- 2 job streams, 4 steps
- Queues at pull operations
- Infinite job supply at 1 and 3
- 2 servers

- Control choice based on $Q_1(t), Q_2(t)$
- No idling, FULL UTILIZATION
- Preemptive resume



Processing Times



$$\xi_k = \{\xi_k^{\ j}, j = 1, 2, \ldots\}, \quad k = 1, 2, 3, 4$$

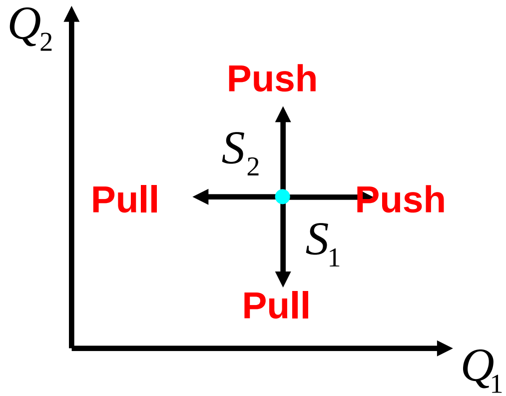
ξ_k i.i.d.

$$E[\xi_1] = 1, \quad E[\xi_3] = 1 \text{ (for simplicity)}$$

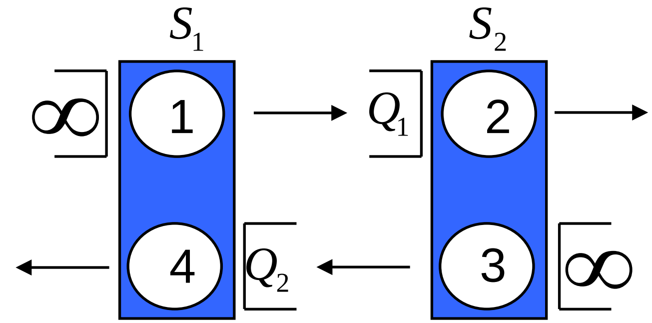
$$E[\xi_2] = \rho_2, \quad E[\xi_4] = \rho_2$$

"interesting" Configurations:

$$\rho_1, \rho_2 < 1 \qquad \text{or} \qquad \rho_1, \rho_2 > 1$$



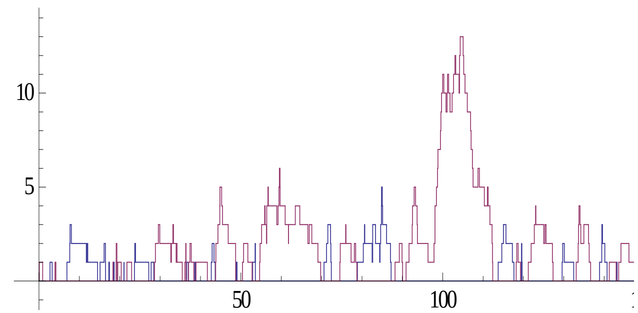
Policies



$$\rho_i < 1$$

Policy: Pull priority (LBFS)

Typical Behavior:

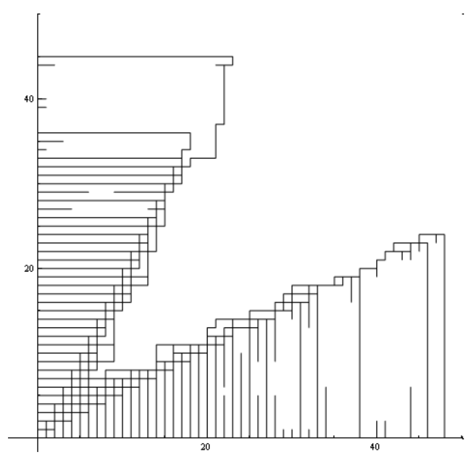


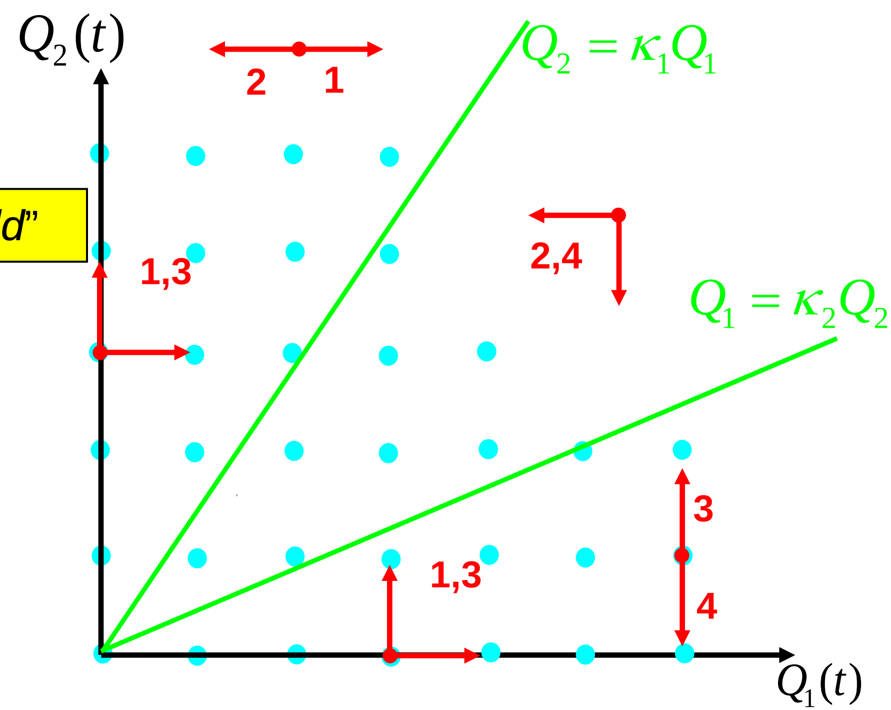
$$\rho_i > 1$$

Policy: Linear thresholds

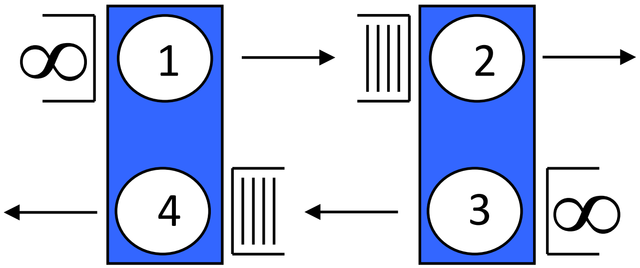
Server: "*don't let opposite queue go below threshold*"

Typical Behavior:





A Markov Process



Queue **Residual**

$$X(t) = \left(\mathrm{Q}(t) \quad \mathrm{U}(t) \right)$$

$X(t)$ is strong Markov with state space $\mathbb{S} = \mathbb{Z}_+^{\ 2} \times \mathbb{R}_+^{\ 2} \times \mathbb{R}_+^{\ 2}$

- For $x \in \mathbb{S}$, $B \in \mathcal{B}(\mathbb{S})$
  $$P^t(x, B) = P_x(X(t) \in B) = P\{X(t) \in B | X(0) = x\}.$$
- π σ-finite on $(\mathbb{S}, B(\mathbb{S}))$ is invariant if for all t
  $$\pi(B) = \int_{\mathbb{S}} P^t(x, B)\, \pi(dx), \quad B \in \mathcal{B}(\mathbb{S}).$$
- $\tau_A = \inf\{t \geq 0 : X(t) \in A\}$. X is Harris recurrent if for measure ν, $A \in \mathcal{B}(\mathbb{S})$ with $\nu(A) > 0$ implies $P_x(\tau_A < \infty) = 1$ for all $x \in \mathbb{S}$.
- Harris Recurrent $\Longrightarrow$ $\exists$ invariant measure π.
- When π is a probability X is positive Harris recurrent.

Push-Pull Results (earlier work)

Stability

Theorem (N., Weiss): Pull-priority, $\rho_i < 1$, $X(t)$ is PHR

Theorem (N., Weiss): Linear thresholds, $\rho_i > 1$, $X(t)$ is PHR

Performance Analysis

Theorem (Kopzon, N., Weiss): Closed form for stationary distribution in specific cases and with memory-less assumptions

Diffusion (CLT) Limits of Outputs

Theorem (N., Weiss): Denote by D(t) the number of outputs during [0,t].

$$D^{(n)}(t) \Rightarrow BM(\Sigma, 0)$$

$$D^{(n)}(t) = \frac{D(nt) - vnt}{\sqrt{n}}$$

Note: This is weak convergence of the sequence $D^{(n)}(t)$ n=1,2,...

Explicit expression for covariance matrix Σ

Main Tool For Stability Results

Establish that an "associated" deterministic system $\overline{X}(t)$ is "stable"

The "framework" then implies that $X(t)$ is "stable"

Nice, since stability of $\overline{X}(t)$ is sometimes easier to establish

This "fluid framework" was pioneered and exploited in 90's by Dai, Meyn, Stoylar, Bramson, Weiss, Chen ….

Stochastic and Fluid Equations

Fluid Network process

$$\bar{Y}(t) = \left(\bar{Q}_1(t),\quad \bar{Q}_2(t),\quad \bar{T}_1(t),\quad \bar{T}_2(t),\quad \bar{T}_3(t),\quad \bar{T}_4(t)\right)$$

Fluid Dynamics

$$\bar{S}_k(t) = \sup\{n : \sum_{j=1}^{n} \xi_k^j \le t\} \; = \mu_k t$$

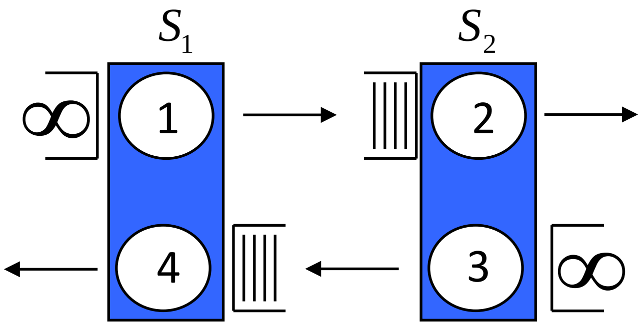
$$\bar{T}_k(0) = 0, \qquad \bar{T}_k(t) \nearrow$$

$$\bar{T}_1(t) + \bar{T}_4(t) = t, \; \bar{T}_2(t) + \bar{T}_3(t) = t$$

$$\bar{D}_k(t) = S_k(T_k(t)) \; = \mu_k \bar{T}_k(t)$$

$$\bar{Q}_k(0) = q_k, \; \bar{Q}_k(t) \ge 0$$

$$\bar{Q}_k(t) = \bar{Q}_k(0) + \bar{D}_{k-1}(t) - \bar{D}_k(t)$$



Pull priority policy

$$\int_0^t \bar{Q}_2(s)\, d\bar{T}_4(s) = 0 \qquad \int_0^t \bar{Q}_1(s)\, d\bar{T}_1(s) = 0$$

E.g. Lyapounov Proofs for Fluid Stability

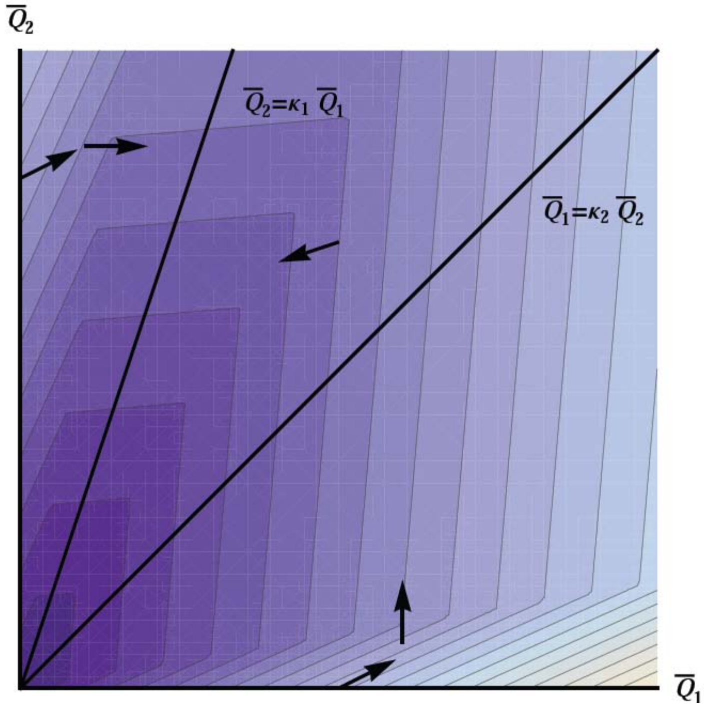
Need: for every solution of fluid model:

- When $f(t)=0$, it stays at 0.
- When $f(t)>0$, at regular points of t, $\dot{f}(t)\leq -\varepsilon$.

$$\rho_i < 1: \quad f(t) = \overline{Q}_2(t) + \overline{Q}_4(t)$$

$$\rho_i > 1:$$

$$f(t) =$$



OUTLINE OF “NEW RESULTS”

For a ring of M servers with Pull-Priority (generalizing Push-Pull)

Theorem (Guo, Lefeber, N., Weiss, Zhang): $X(t)$ is PHR (stable) if

(i) $\rho_i < 1 \;\; i = 1, \dots, M$

(ii) When M is odd and $1 < \rho_i \;\; i = 1, \dots, M$ and

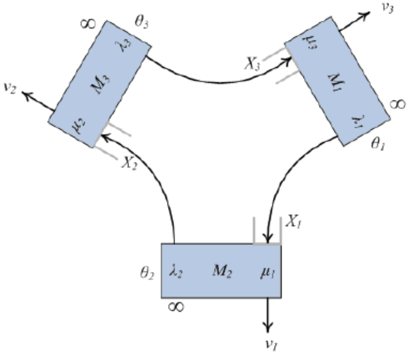
$$\Delta = \sum_{i=1}^{M} c_i \left(\frac{\mathrm{M}-1}{2} (\rho_i - 1) - 1 \right) < 0$$

$$c_i = (\dots(\rho_{i-1} - 1)\rho_{i-2} + 1)\rho_{i-3} - 1)\rho_{i-4} + \cdots + (-1)^{1+(M-1)/2})\rho_{i+1} + (-1)^{(M-1)/2}$$

Specifically when $\rho_i = \rho$ then Δ<0 iff $\rho < 1 + \frac{2}{M-1}$

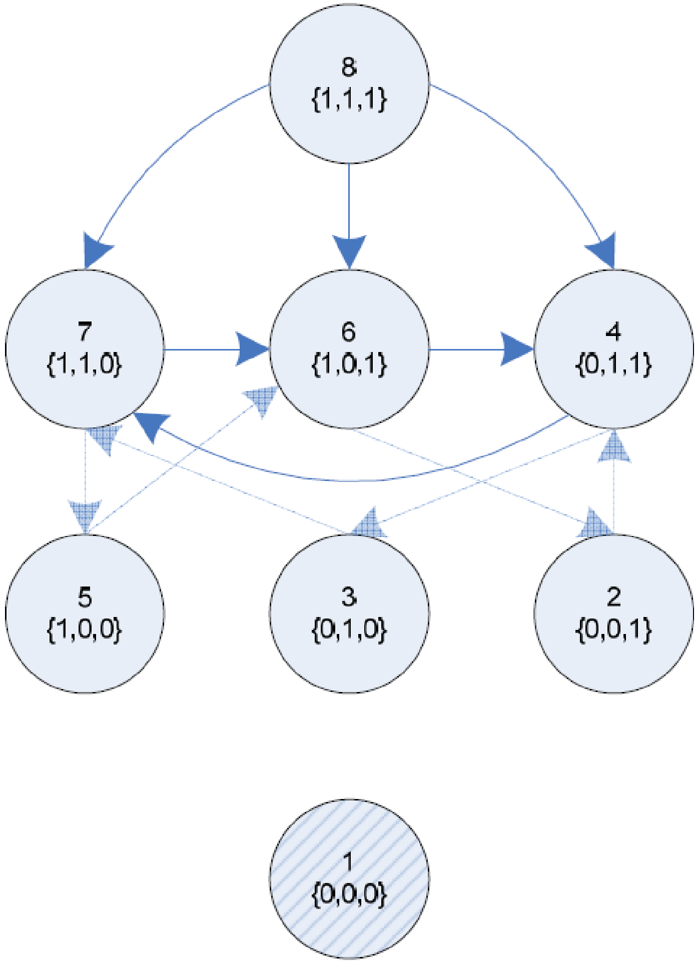
And when M=3,

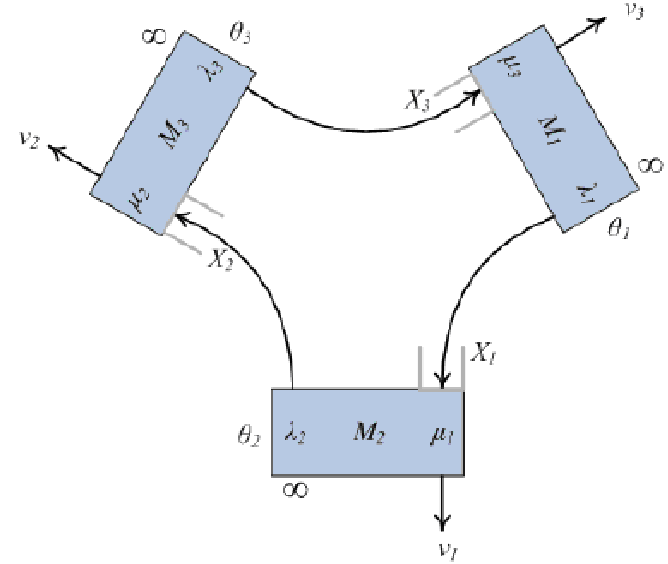
$$\Delta = \prod_{i=1}^{3} (\rho_i - 1) - 1$$

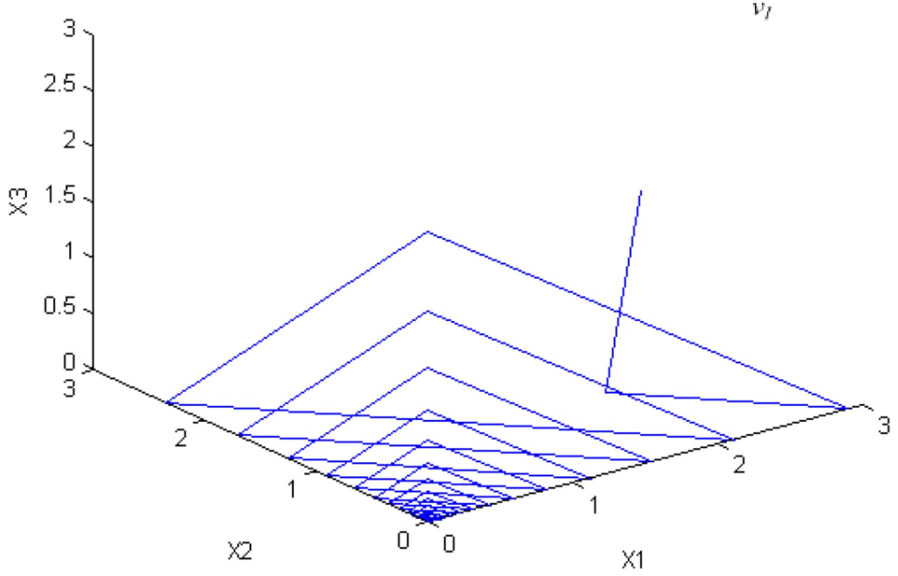


Stable Fluid Trajectory of M=3 Pull-Priority

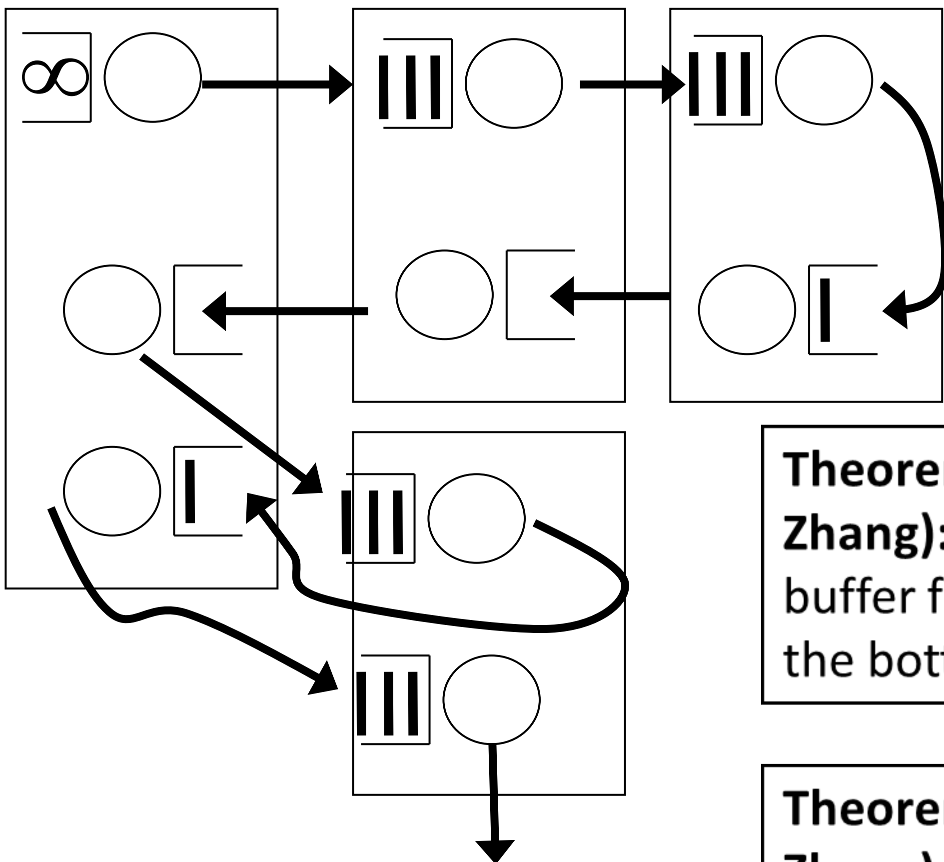
$\rho_i > 1$







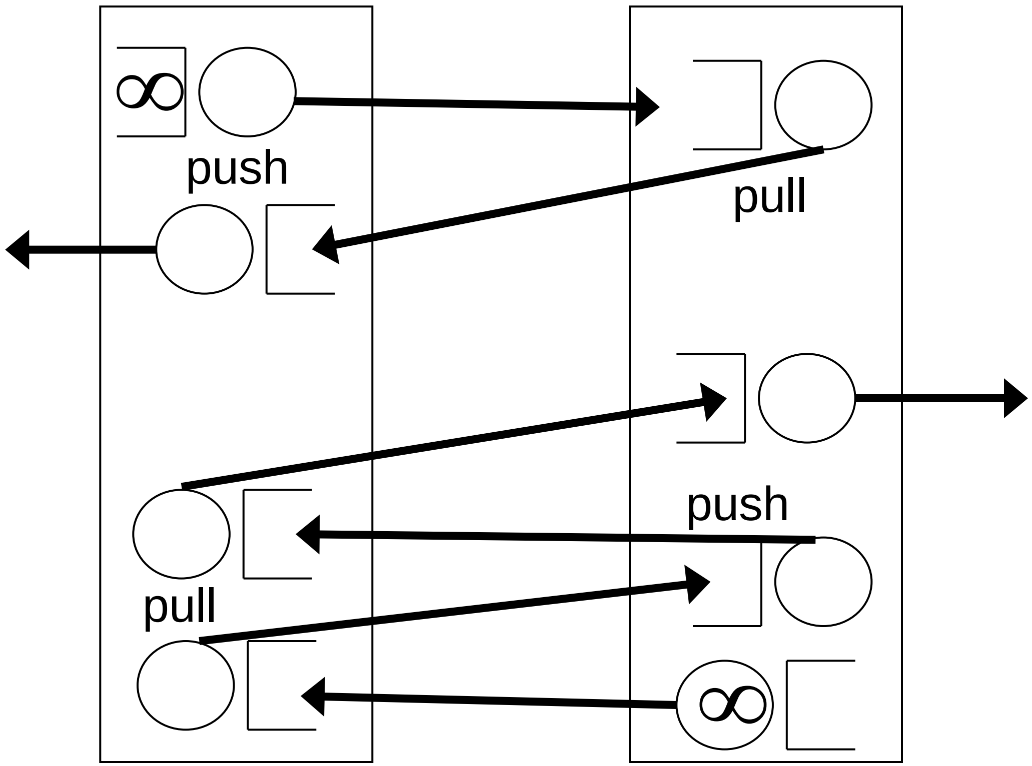
A re-entrant line with infinite supply



Theorem (Guo, Lefeber, N., Weiss, Zhang): $X(t)$ is PHR (stable) under the last-buffer first served (LBFS) policy if first server is the bottleneck.

Theorem (Guo, Lefeber, N., Weiss, Zhang): $X(t)$ is PHR (stable) under the FBFS policy under further conditions.

Two re-entrant lines in a push-pull



Theorem (Guo, Lefeber, N., Weiss, Zhang):
$\boldsymbol{X}(\boldsymbol{t})$ is PHR (stable) under a “pull-LBFS” priority policy if
push activities mean > pull activities mean
in each of the re-entrant lines

Diffusion Limit Results

Theorem (Guo, Lefeber, N., Weiss, Zhang):
For general Multi-Class Queueing Networks with Infinite Supplies and determinstic routes (as in this talk), if the network is PHR,

$$D^{(n)}(t) \Rightarrow BM(\Sigma, 0)$$

With explicit "matrix expressions" for Σ.

Note that the diffusion limit does not depend on the specific policy (as long as it is a stabalizing policy).

$$D^{(n)}(t) = \frac{D(nt) - vnt}{\sqrt{n}}$$

Summary

- Specific cases of networks with infinite supplies (and full utilization) can be analysed with “some effort”

- General policies for stabilizing general networks remains an “open problem”

Recommended book on the subject: Bramson, 2009, Stability of queueing networks.

Association of Fluid Model and Stochastic System

fluid scalings

$$\overline{Y}^n(t,\omega) = \frac{Y^n(nt,\omega)}{n}$$

$\overline{Y}(t) = \left(\overline{Q}(t) \quad \overline{T}(t)\right)$ is fluid limit

if exists $r \to \infty$ and ω: $\overline{\mathrm{Y}}^{\mathrm{r}}(\cdot,\omega) \to \overline{Y}(\cdot)$, u.o.c.

$\overline{Y}$ is associated with Y

if w.p.1 every fluid limit is a fluid model solution

Stability of Fluid Model

Definition: A fluid model is stable, if when ever, $q_1 + q_2 = 1$
there exists T, such that for all solutions,

$$Q_1(t) + Q_2(t) = 0 \quad \forall \mathrm{t} \geq \mathrm{T}$$

Definition: A fluid model is weakly stable, if when ever $q_1 + q_2 = 0$

$$Q_1(t) + Q_2(t) = 0 \quad \forall \mathrm{t} \geq 0$$

Main Results of "Fluid Limit Method"