Scaling Limits, Cyclically Varying Birth-Death Processes and Stationary Distributions

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Overview

- Birth death processes
- Scaling limits
- Cyclically varying systems
- Stationary distributions
Talk Outline

- Part 1: Approximating trajectories (of birth-death processes)

- Part 2: Approximating stationary distributions (of cyclic processes)
Part 1: Approximating Trajectories
An Example Class of Birth Death Processes

- \( \{X(t), t \geq 0\} \) is a Continuous Time, Birth-Death, Markov Chain taking values \( \{0, 1, \ldots\} \)
- Birth rates are constant: \( \lambda \)
- Death rates are state dependent: \( \mu X(t)^{\alpha}, \; \alpha \geq 0 \)
- \( \alpha = 0 \) is M/M/1, \( \alpha = 1 \) is M/M/\( \infty \)

Desired: A deterministic \( x(t) \) that approximates \( X(t) \)

Scaling The Processes

A sequence of processes

- $X_N(\cdot)$, $N = 1, 2, \ldots$
- The parameters of the $N$'th process: $\lambda_N, \mu_N$ and $\alpha$
- Initial values are $X_N(0) = N X(0)$
- Desired: $X_N(t) \approx N x(t)$ as $N \rightarrow \infty$ (for finite $t$)

Try $x(t)$, solution of the ODE:

$$\dot{x}(t) = \lambda - \mu x(t)^\alpha$$
$$x(0) = X(0)$$

What is a "correct" scaling?
Scaling The Processes

A sequence of processes
- \( X_N(\cdot), \; N = 1, 2, \ldots \)
- The parameters of the \( N \)'th process: \( \lambda_N, \mu_N \) and \( \alpha \)
- Initial values are \( X_N(0) = N X(0) \)
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Try \( x(t) \), solution of the ODE:
\[
\dot{x}(t) = \lambda - \mu x(t)^\alpha \\
x(0) = X(0)
\]

What is a ”correct” scaling?

Observe from the ODE:
\[
\lambda_N = \lambda N, \; \mu_N = \mu N^{1-\alpha}
\]
Illustration for $\alpha = \frac{2}{3}$

$N = 1$

$\lambda = \mu = 1, \quad X(0) = 5$
Illustration for $\alpha = \frac{2}{3}$

$N = 10$

$\lambda = \mu = 1$, $X(0) = 5$
Illustration for $\alpha = 2/3$

$N = 100$

$\lambda = \mu = 1, \quad X(0) = 5$
Illustration for $\alpha = \frac{2}{3}$

$N = 1000$

$\lambda = \mu = 1, \quad X(0) = 5$
Theorem

(i) Trajectories:

\[
\lim_{N \to \infty} P \left( \sup_{s \in [0, t]} \left| \frac{X_N(s)}{N} - x(s) \right| > \epsilon \right) = 0
\]

(ii) Hitting Times:

\[
\lim_{N \to \infty} P \left( \left| T_N(yN) - \tau(y) \right| > \epsilon \right) = 0
\]

where,

\[T_N(y) = \inf \{ t : X_N(t) = y \}, \quad \tau(y) = \inf \{ t : x(t) = y \} = x^{-1}(y)\]

Note 1: For \( \alpha = 0, 1 \) it is well known, see P. Robert book, 2003

Note 2: Also have formulation for more general BD processes
Martingale Representation

\[ X_N(t) = X_N(0) + M_N(t) + \lambda_N t - \mu_N \int_0^t X_N(s)^\alpha ds \]

Substitute: \( X_N(0) = N X(0), \lambda_N = \lambda N, \mu_N = \mu N^{1-\alpha} \) and divide by \( N \)

\[ \frac{X_N(t)}{N} = X(0) + \frac{M_N(t)}{N} + \lambda t - \mu \int_0^t \left( \frac{X_N(s)}{N} \right)^\alpha ds \]

Compare With the Deterministic Trajectory:

\[ x(t) = X(0) + \lambda t - \mu \int_0^t x(s)^\alpha ds \]

\[ \sup_{s \in [0,t]} \left| \frac{X_N(s)}{N} - x(s) \right| \leq \sup_{s \in [0,t]} \left| \frac{M_N(s)}{N} \right| + \int_0^t \sup_{u \in [0,s]} \left| \left( \frac{X_N(u)}{N} \right)^\alpha - x(u)^\alpha \right| ds \]
Part 2: Approximating Stationary Distributions (of Cyclically Varying Systems)
Cyclically Varying Systems

- A sequence of increasing time points \( \{ T_n, n \geq 0 \} \)
- Two sets of birth-death parameters \( \Lambda_i = (\lambda_i, \mu_i), \ i = 1, 2 \)
- At time points \( T_n \), \( X(t) \) changes behavior, alternating between \( \Lambda_1 \) and \( \Lambda_2 \)
Types of Cyclic Behavior

Hysteresis Control

\[ T_n = \inf\{ t > T_{n-1} : X(t) = \begin{cases} \ell_2 & n \text{ odd} \\ \ell_1 & n \text{ even} \end{cases} \} \]

Fixed Cycles

\[ T_n - T_{n-1} = \begin{cases} \tau_1 & n \text{ odd} \\ \tau_2 & n \text{ even} \end{cases} \]

Random Environment

\[ T_n - T_{n-1} \sim \begin{cases} \exp(\tau_1^{-1}) & n \text{ odd} \\ \exp(\tau_2^{-1}) & n \text{ even} \end{cases} \]
## Some of the Related Literature

<table>
<thead>
<tr>
<th><strong>Hysteresis Control</strong></th>
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<td>Federgruen and Tijms 1980, Perry 1997, Bekker 2009...</td>
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In general, the queue level distribution is ”tough”. Things get ”tougher” as one moves from $\alpha = 0$ to $\alpha = 1$ and then to arbitrary $\alpha$. 
Basic Idea: Use the Scaling Limits

Random Environment

Hysteresis Control

Fixed Cycles
Basic Idea: Use the Scaling Limits

Hysteresis Control
Look at one deterministic cycle through $\ell_1 \rightarrow \ell_2 \rightarrow \ell_1$

Fixed Cycles
Look at one deterministic cycle of duration $\tau_1 + \tau_2$

Random Environment
Look at a piece-wise deterministic Markov process (PDMP)

In all 3 cases: Construct a distribution function $F(\cdot)$ by means of the scaling limit
\( F(\cdot) \) for Hysteresis Control and Fixed Cycles

\[
\dot{x}_i(t) = \lambda_i - \mu_i x(t)^\alpha \\
 x_i(0) = \ell_i \\
\lim_{t \to \infty} x_i(t) = m_i \\
m_2 < \ell_1 < \ell_2 < m_1 \\
\dot{x}_2(0) < 0 < \dot{x}_1(0) \\
\tau_i(y) = \inf \{t : x_i(t) = y\} \\
\tau_i = \tau_i(\ell_i)
\]

A CDF with support \([\ell_1, \ell_2]\), (assume \( \alpha > 0 \))

\[
F(y) = \frac{1}{\tau_1 + \tau_2} (\tau_1(y) + (\tau_2 - \tau_2(y))
\]

- For Hysteresis control, \( \ell_1, \ell_2 \) given, \( \tau_1, \tau_2 \) easily calculated
- For Fixed Cycles \( \tau_1, \tau_2 \) given, unique \( \ell_1, \ell_2 \) obtained by solving:

\[
\begin{align*}
\left. x_1 \right|_{x_1(0) = \ell_1}^{(\tau_1)} &= \ell_2, \\
\left. x_2 \right|_{x_2(0) = \ell_2}^{(\tau_2)} &= \ell_1
\end{align*}
\]
**$F(\cdot)$ for Random Environment**

- **PDMP**: Environment Markov chain alternates between 1, 2. Given a mode, trajectory is deterministic with "state-dependent" rates.

**Stationary Distribution**

Solve for $p_1(\cdot), p_2(\cdot)$ on $y \in (m_2, m_1)$

\[
(\lambda_1 - \mu_1 y^\alpha) p'_1(y) = \tau_2^{-1} p_2(y) - \tau_1^{-1} p_1(y) \\
(\lambda_2 - \mu_2 y^\alpha) p'_2(y) = \tau_1^{-1} p_1(y) - \tau_2^{-1} p_2(y)
\]

\[ p_1(m_2) = 0, \quad p_2(m_1) = \frac{\tau_2}{\tau_1 + \tau_2} \]

\[ F(y) = p_1(y) + p_2(y), \quad y \in (m_2, m_1) \]
Some Cases where $F(\cdot)$ is explicit

Hysteresis Control or Fixed Cycles where $\alpha = 1$

\[
F(y) = \int_{-\infty}^{y} f(u) du, \quad f(u) = \frac{(\mu_1 - \mu_2)u + (\lambda_2 - \lambda_1)}{(\mu_1 u - \lambda_1)(\mu_2 u - \lambda_2)} \log \left( \frac{\mu_1 \ell_1 - \lambda_1}{\mu_1 \ell_2 - \lambda_1} \right) \frac{1}{\mu_1} \left( \frac{\mu_2 \ell_2 - \lambda_2}{\mu_2 \ell_1 - \lambda_2} \right) \frac{1}{\mu_2} 1\{\ell_1 \leq u \leq \ell_2\}
\]

For fixed cycles set: $\ell_i = \frac{(e^{\tau_i \mu_i} - 1) \frac{\lambda_i}{\mu_i} + (e^{\tau_i \mu_i} - 1) \frac{\lambda_i}{\mu_i} e^{\tau_i \mu_i}}{e^{\tau_i \mu_i} + \tau_i \mu_i - 1}$

Hysteresis Control or Fixed Cycles with $\alpha = 0$

Uniform distribution, sometimes with masses at the endpoints

Random Environment with $\alpha = 0$

Truncated exponential distribution with masses at $m_1$ and $m_2$

Random Environment with $\alpha = 1$

When $\mu_1 = \mu_2 = \tau_1 = \tau_2 = 1$, uniform on $[\lambda_2, \lambda_1]$. Otherwise, more complex explicit expression
Assume \( X_N(\cdot) \) is positive-recurrent. Then,

\[
\lim_{N \to \infty} \sup_{y} \left| P\left( \frac{X_N(\infty)}{N} \leq y \right) - F(y) \right| = 0,
\]

In the hysteresis control case, also scale the thresholds: \( \left( [N\ell_1], [N\ell_2] \right) \)
Numerical Example: Hysteresis Control and Fixed Cycles

\[ \alpha = 1 \quad \mu_1 = \mu_2 = 1 \quad \lambda_1 = 2 \quad \lambda_2 = 0.2 \]
\[ \ell_1 = 0.3 \quad \ell_2 = 1.6 \quad \tau_1 = 1.447 \quad \tau_2 = 2.639 \]
 Numerical Example: Hysteresis Control and Fixed Cycles

\[ \alpha = 1 \quad \mu_1 = \mu_2 = 1 \quad \lambda_1 = 2 \quad \lambda_2 = 0.2 \]
\[ \ell_1 = 0.3 \quad \ell_2 = 1.6 \quad \tau_1 = 1.447 \quad \tau_2 = 2.639 \]

N = 5

Limit \( F(y) \)

Hysteresis Control
\( P(X_N(\infty)/N \leq y) \)

Fixed Cycles
\( P(X_N(\infty)/N \leq y) \)
Numerical Example: Hysteresis Control and Fixed Cycles

\[ \alpha = 1 \quad \mu_1 = \mu_2 = 1 \quad \lambda_1 = 2 \quad \lambda_2 = 0.2 \]
\[ \ell_1 = 0.3 \quad \ell_2 = 1.6 \quad \tau_1 = 1.447 \quad \tau_2 = 2.639 \]

\[ N = 10 \]
Numerical Example: Hysteresis Control and Fixed Cycles

\[
\begin{align*}
\alpha &= 1 & \mu_1 &= \mu_2 = 1 & \lambda_1 &= 2 & \lambda_2 &= 0.2 \\
\ell_1 &= 0.3 & \ell_2 &= 1.6 & \tau_1 &= 1.447 & \tau_2 &= 2.639
\end{align*}
\]

\[N = 50\]
Numerical Example: Hysteresis Control and Fixed Cycles

\[ \alpha = 1 \quad \mu_1 = \mu_2 = 1 \quad \lambda_1 = 2 \quad \lambda_2 = 0.2 \]
\[ \ell_1 = 0.3 \quad \ell_2 = 1.6 \quad \tau_1 = 1.447 \quad \tau_2 = 2.639 \]

\[ N = 100 \]
Numerical Example: Hysteresis Control and Fixed Cycles

\[ \alpha = 1 \quad \mu_1 = \mu_2 = 1 \quad \lambda_1 = 2 \quad \lambda_2 = 0.2 \]

\[ \ell_1 = 0.3 \quad \ell_2 = 1.6 \quad \tau_1 = 1.447 \quad \tau_2 = 2.639 \]

\[ N = 500 \]
Numerical Example: Random Environment - Uniform

\[ \alpha = 1 \]

\[ \mu_1 = \mu_2 = \tau_1 = \tau_2 = 1, \quad \lambda_1 = 3, \quad \lambda_2 = 1, \]

\[ N = 1, 10, 100: \]
Numerical Example: Random Environment

\[ \alpha = 4/3 \]

\[ \mu_1 = \mu_2 = 1, \ \lambda_1 = 2, \ \lambda_2 = 1/2, \ \tau_1 = 3, \ \tau_2 = 1 \]

\[ N = 50, 100, 500, 1000: \]
Questions?