Scaling limits of cyclically varying birth-death processes

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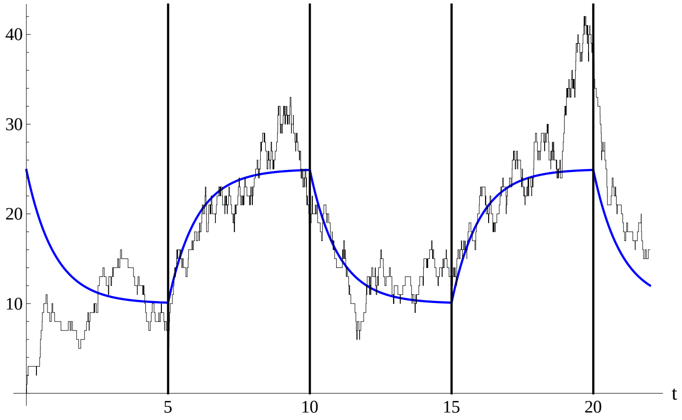
* The research began while both authors were at both at TU Eindhoven / EURANDOM and CWI Amsterdam

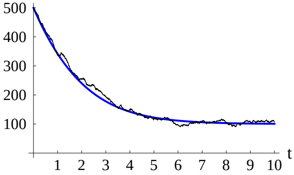
Overview

- Birth death processes
- Scaling limits
- Cyclically varying systems
- Stationary distributions

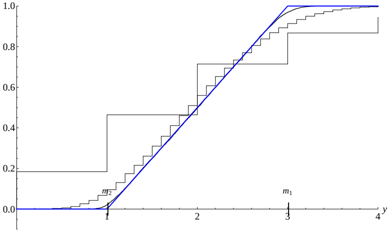


Talk Outline

- Part 1: Approximating trajectories (of birth-death processes)



- Part 2: Approximating stationary distributions (of cyclic processes)



Part 1: Approximating Trajectories

An Example Class of Birth Death Processes

- $\{X(t), t \geq 0\}$ is a Continuous Time, Birth-Death, Markov Chain taking values $\{0, 1, \ldots\}$
- Birth rates are constant: λ
- Death rates are state dependent: $\mu \cdot \Big(X(t)\Big)^{\alpha}, \;\; \alpha \geq 0$
- $\alpha = 0$ is M/M/1, $\alpha = 1$ is M/M/∞

Desired: A deterministic $x(t)$ that approximates $X(t)$

Some ideas: R.W.R. Darling, J.R. Norris, *Differential equation approximations for Markov chains*, Probability Surveys, 5, pp. 37-79, 2008.

Scaling The Processes

A sequence of processes

- $X_N(\cdot)$, $N = 1, 2, \ldots$
- The parameters of the N'th process: λ_N, μ_N and α
- Initial values are $X_N(0) = N \cdot X(0)$
- Desired: $X_N(t) \approx N\,x(t)$ as $N \to \infty$ (for finite t)

Try $x(t)$, solution of the ODE:

$$\begin{aligned} \dot{x}(t) &= \lambda - \mu\big(x(t)\big)^{\alpha} \\ x(0) &= X(0) \end{aligned}$$

What is a "correct" scaling for λ_N, μ_N?

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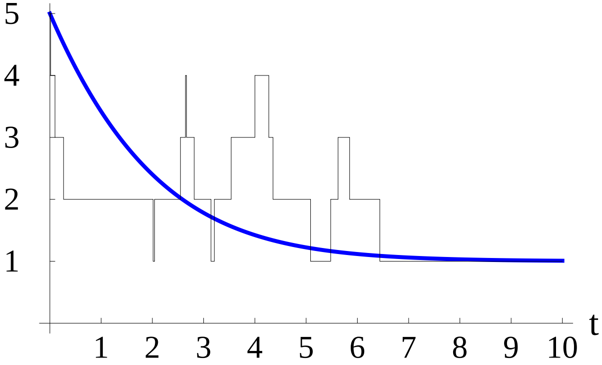
Answer:

$$\lambda_N = \lambda N, \mu_N = \mu N^{1-\alpha}$$

Illustration for $\alpha = 2/3$

$N = 1$



$\lambda = \mu = 1, \quad X(0) = 5$

Illustration for $\alpha = 2/3$

$N = 10$

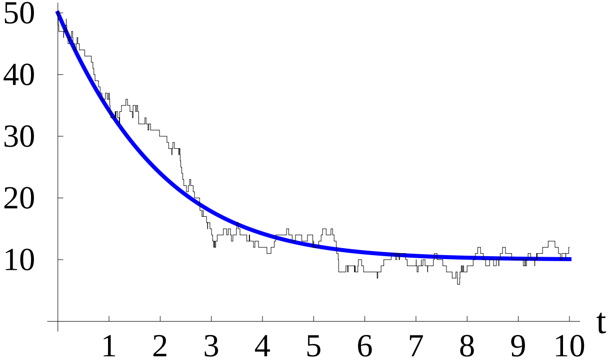


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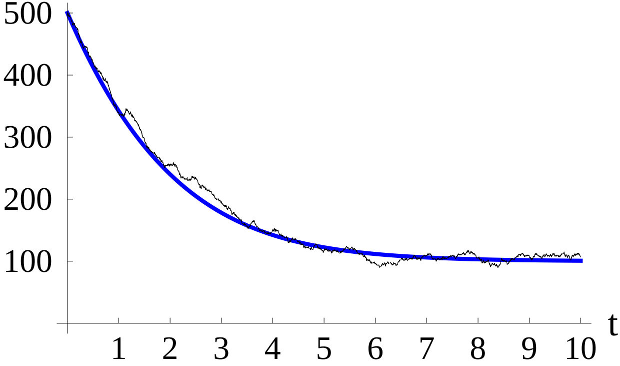
Illustration for $\alpha = 2/3$

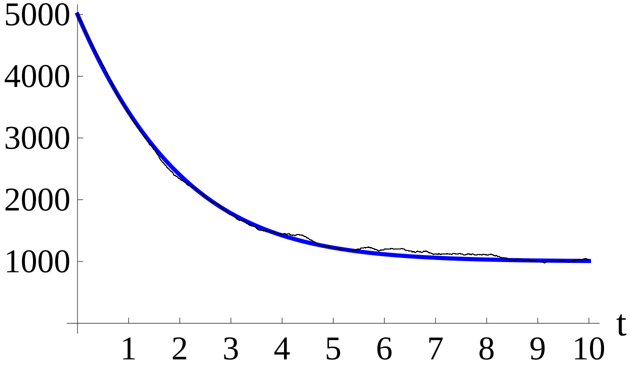
$N = 100$



$\lambda = \mu = 1, \quad X(0) = 5$

Illustration for $\alpha = 2/3$

$$N = 1000$$



$\lambda = \mu = 1, \quad X(0) = 5$

Theorem

(i) Trajectories:

$$\lim_{N\to\infty} P\left(\sup_{s\in[0,t]} \left| \frac{X_N(s)}{N} - x(s) \right| > \epsilon \right) = 0$$

(ii) Hitting Times:

$$\lim_{N\to\infty} P\left(\left| \mathcal{T}_N(yN) - \tau(y) \right| > \epsilon \right) = 0$$

where,

$$\mathcal{T}_N(y) = \inf\{t : X_N(t) = y\}, \quad \tau(y) = \inf\{t : x(t) = y\} = x^{-1}(y)$$

Note 1: For $\alpha = 0, 1$ result is well known
(there are even a.s. and L_1 versions)

Note 2: We have a formulation for more general BD processes
(the choice of rates λ and $\mu\big(X(t)\big)^{\alpha}$ is for presentation simplicity)

Martingale Representation

$$X_N(t) = X_N(0) + M_N(t) + \lambda_N t - \mu_N \int_0^t \big(X_N(s)\big)^\alpha ds$$

Substitute: $X_N(0) = N\,X(0)$, $\lambda_N = \lambda N$, $\mu_N = \mu N^{1-\alpha}$ and Divide by N

$$\frac{X_N(t)}{N} = X(0) + \frac{M_N(t)}{N} + \lambda t - \mu \int_0^t \Big(\frac{X_N(s)}{N}\Big)^\alpha ds$$

Compare With the Deterministic Trajectory:

$$x(t) = X(0) + \lambda t - \mu \int_0^t \big(x(s)\big)^\alpha ds$$

$$sup_{s\in[0,t]}\Big|\frac{X_N(s)}{N} - x(s)\Big| \le sup_{s\in[0,t]}\frac{|M_N(s)|}{N} + \mu \int_0^t sup_{u\in[0,s]}\Big|\Big(\frac{X_N(u)}{N}\Big)^\alpha - \big(x(u)\big)^\alpha\Big| ds$$

$$sup_{s\in[0,t]}\left|\frac{X_N(s)}{N}-x(s)\right| \leq sup_{s\in[0,t]}\frac{\left|M_N(s)\right|}{N}+\mu\int_0^t sup_{u\in[0,s]}\left|\left(\frac{X_N(u)}{N}\right)^{\alpha}-\left(x(u)\right)^{\alpha}\right|ds$$

Gronwall's lemma makes things easy for $\alpha = 1$:

$$\epsilon(t) \leq m(t) + \mu\int_0^t \epsilon(s)ds \quad \Rightarrow \quad \epsilon(t) \leq m(t)e^{\mu t}$$

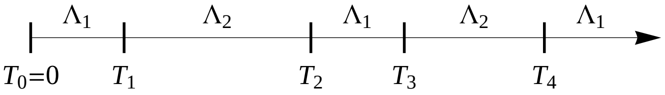
And then $m(t) = sup_{s\in[0,t]}\frac{\left|M_N(s)\right|}{N}$ "vanishes" as $N \to \infty$

The above is "standard", see for example: Philippe Robert 2003, *Stochastic Networks and Queues*. But in case of $\alpha \neq 1$ we need to work harder: Control the probability of being in a set where Gronwall can be applied.

Part 2: Approximating Stationary Distributions
(of Cyclically Varying Systems)

Cyclically Varying Systems

- A sequence of increasing time points $\{T_n, n \geq 0\}$
- Two sets of birth-death parameters $\Lambda_i = \left(\lambda_i, \mu_i\right)$, $i = 1, 2$
- At time points T_n, $X(t)$ changes behavior, alternating between Λ_1 and Λ_2



Types of Cyclic Behavior

Hysteresis Control

$$T_n = \inf\{t > T_{n-1} : X(t) = \begin{cases} \ell_2 & n \text{ odd} \\ \ell_1 & n \text{ even} \end{cases} \}$$

Fixed Cycles

$$T_n - T_{n-1} = \begin{cases} \tau_1 & n \text{ odd} \\ \tau_2 & n \text{ even} \end{cases}$$

Random Environment

$$T_n - T_{n-1} \sim \begin{cases} \exp\left(\tau_1^{-1}\right) & n \text{ odd} \\ \exp\left(\tau_2^{-1}\right) & n \text{ even} \end{cases}$$

Some of the Related Literature

Hysteresis Control

Federgruen and Tijms 1980, Perry 1997, Bekker 2009...

Fixed Cycles

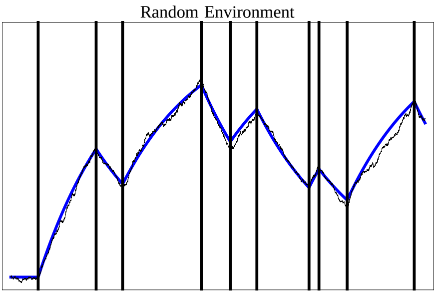
Harrison and Lemoine 1977, Lemoine 1989, Breuer 2004, Mandjes 2011...

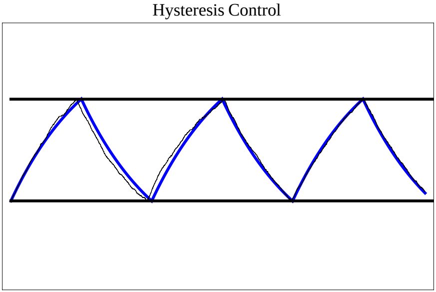
Random Environment

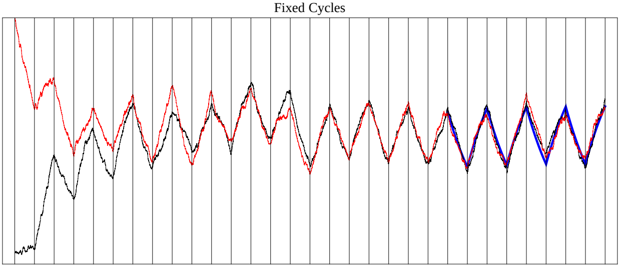
Yechiali and Naor 1971, Neuts 1977, Prabhu and Zhu 1989, Boxma and Kurkova 2000, Falin 2008, Fralix and Adan 2009, Mandjes 2011...

In general, the queue level distribution is "tough". Things get "tougher" as one moves from $\alpha = 0$ to $\alpha = 1$ and then to arbitrary α.

Basic Idea: Use the Scaling Limits







Basic Idea: Use the Scaling Limits

Hysteresis Control

Look at one deterministic cycle through $\ell_1 \to \ell_2 \to \ell_1$

Fixed Cycles

Look at one deterministic cycle of duration $\tau_1 + \tau_2$

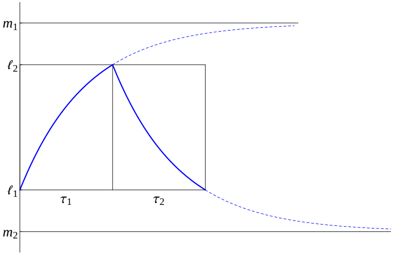
Random Environment

Look at a piece-wise deterministic Markov process (PDMP)

In all three cases we propose a distribution function $F(\cdot)$ motivated by the scaling limit

$F(\cdot)$ for Hysteresis Control and Fixed Cycles

$\dot{x}_i(t) = \lambda_i - \mu_i\big(x(t)\big)^{\alpha}$
$x_i(0) = \ell_i$
$\lim_{t\to\infty} x_i(t) = m_i$
$m_2 < \ell_1 < \ell_2 < m_1$
$\dot{x}_2(0) < 0 < \dot{x}_1(0)$
$\tau_i(y) = \inf\{t : x_i(t) = y\}$
$\tau_i = \tau_i(\ell_{\bar{i}})$



A CDF with support $[\ell_1, \ell_2]$, (assume $\alpha > 0$)

$$F(y) = \frac{1}{\tau_1 + \tau_2}\big(\tau_1(y) + (\tau_2 - \tau_2(y))\big)$$

- For Hysteresis control, ℓ_1, ℓ_2 given, τ_1, τ_2 easily calculated
- For Fixed Cycles τ_1, τ_2 given, unique ℓ_1, ℓ_2 obtained by solving:

$$x_1\Big|_{x_1(0)=\ell_1}^{(\tau_1)} = \ell_2, \qquad x_2\Big|_{x_2(0)=\ell_2}^{(\tau_2)} = \ell_1$$

$F(\cdot)$ for Random Environment

- PDMP: Environment Markov chain alternates between $1, 2$. Given a mode, trajectory is deterministic with "state-dependent" rates
- O. Kella and W. Stadje, *Exact Results for a Fluid Model with State-Dependent Flow Rates*, Prob. in Eng. and Inform. Sci., 16, pp. 389-402, 2002

Stationary Distribution

Solve for $p_1(\cdot), p_2(\cdot)$ on $y \in (m_2, m_1)$

$$\begin{aligned} (\lambda_1 - \mu_1 y^{\alpha}) p_1'(y) &= {\tau_2}^{-1} p_2(y) - {\tau_1}^{-1} p_1(y) \\ (\lambda_2 - \mu_2 y^{\alpha}) p_2'(y) &= {\tau_1}^{-1} p_1(y) - {\tau_2}^{-1} p_2(y) \\ p_1(m_2) = 0, &\qquad p_2(m_1) = \frac{\tau_2}{\tau_1 + \tau_2} \end{aligned}$$

$$F(y) = p_1(y) + p_2(y), \quad y \in (m_2, m_1)$$

Some Cases where $F(\cdot)$ is explicit

Hysteresis Control or Fixed Cycles where $\alpha = 1$

$$F(y) = \int_{-\infty}^{y} f(u)du, \qquad f(u) = \frac{\frac{(\mu_1-\mu_2)u+(\lambda_2-\lambda_1)}{(\mu_1 u-\lambda_1)(\mu_2 u-\lambda_2)}}{\log\left(\frac{\mu_1\ell_1-\lambda_1}{\mu_1\ell_2-\lambda_1}\right)^{\frac{1}{\mu_1}}\left(\frac{\mu_2\ell_2-\lambda_2}{\mu_2\ell_1-\lambda_2}\right)^{\frac{1}{\mu_2}}}\mathbf{1}_{\{\ell_1\le u\le\ell_2\}}$$

For fixed cycles set: $\ell_i = \frac{(e^{\tau_i\mu_i}-1)\frac{\lambda_i}{\mu_i}+(e^{\tau_{\bar{i}}\mu_{\bar{i}}}-1)\frac{\lambda_{\bar{i}}}{\mu_{\bar{i}}}e^{\tau_i\mu_i}}{e^{\tau_i\mu_i+\tau_{\bar{i}}\mu_{\bar{i}}}-1}$

Hysteresis Control or Fixed Cycles with $\alpha = 0$

Uniform distribution, sometimes with masses at the endpoints

Random Environment with $\alpha = 0$

Truncated exponential distribution with masses at m_1 and m_2

Random Environment with $\alpha = 1$

When $\mu_1 = \mu_2 = \tau_1 = \tau_2 = 1$, uniform on $[\lambda_2, \lambda_1]$. Otherwise, more complex explicit expression

Convergence of Stationary Distributions

Let $X_N(\cdot)$ be the scaled modulated process. Assume it is positive-recurrent. Then:

$$\lim_{N\to\infty} P\big(\frac{X_N(\infty)}{N} \leq y\big) = F(y),$$

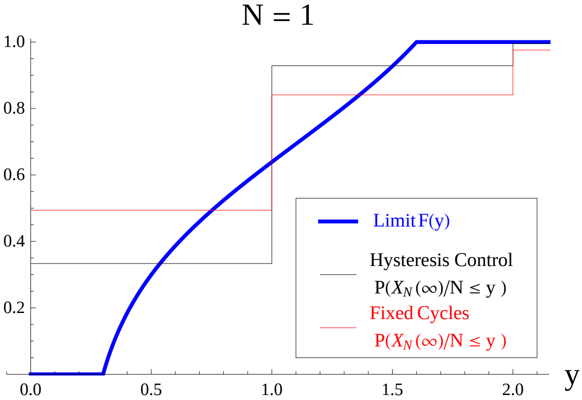
for y where $F(\cdot)$ is continuous.

Note: For the N'th hysteresis control system use thresholds $\Big(\lceil N\ell_1\rceil, \lfloor N\ell_2\rfloor\Big)$

Proof for hysteresis control is "easy", proof for random environment requires "more care" and IS NOT YET COMPLETE

Numerical Example: Hysteresis Control and Fixed Cycles

$$\alpha = 1 \quad \mu_1 = \mu_2 = 1 \quad \lambda_1 = 2 \quad \lambda_2 = 0.2$$
$$\ell_1 = 0.3 \quad \ell_2 = 1.6 \quad \tau_1 = 1.447 \quad \tau_2 = 2.639$$



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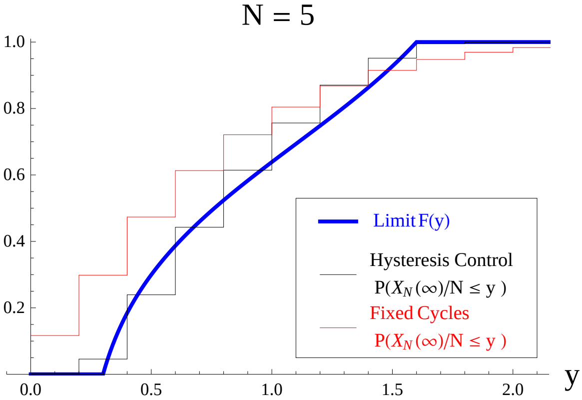

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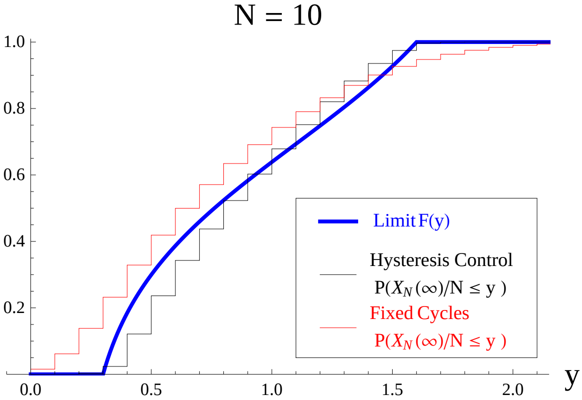

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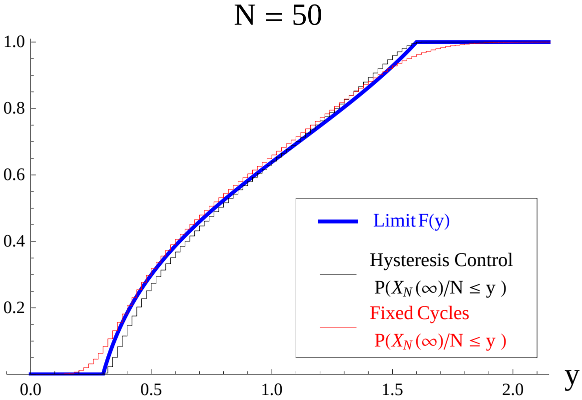

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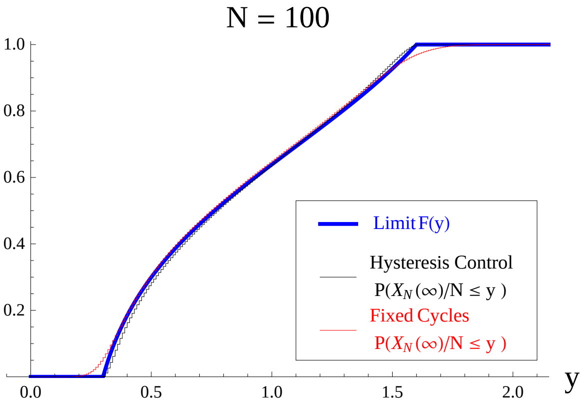

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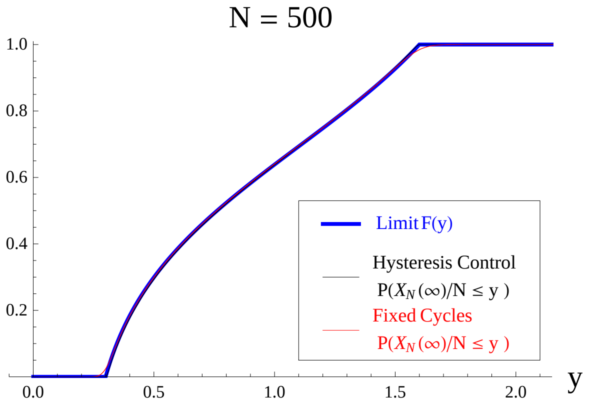
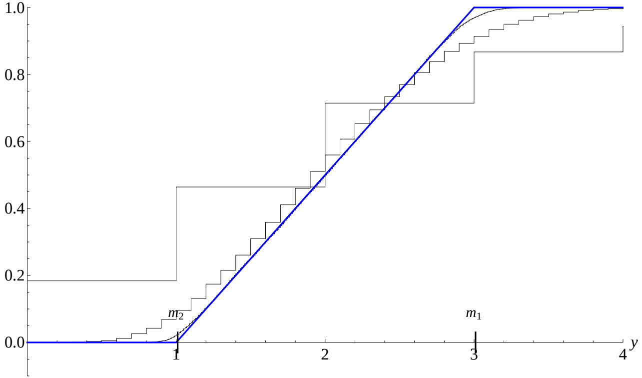


Numerical Example: Random Environment - Uniform

$$\alpha = 1$$

$$\mu_1 = \mu_2 = \tau_1 = \tau_2 = 1, \ \lambda_1 = 3, \ \lambda_2 = 1,$$

$$N = 1, 10, 100:$$

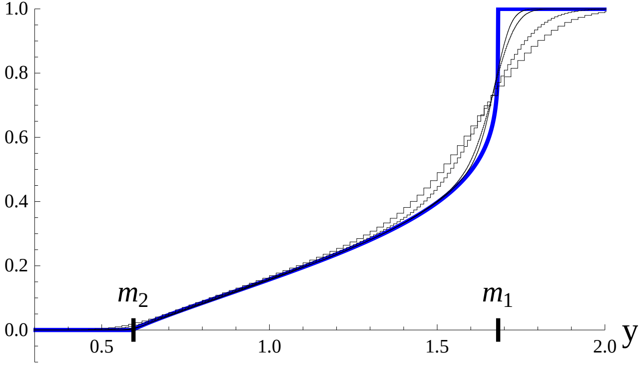


Numerical Example: Random Environment

$$\alpha = 4/3$$

$$\mu_1 = \mu_2 = 1,\ \lambda_1 = 2,\ \lambda_2 = 1/2,\ \tau_1 = 3,\ \tau_2 = 1$$

$$N = 50, 100, 500, 1000:$$



Conclusion

- Scaling (fluid) limits in queueing theory is not a new concept
- Contribution 1: Scaling limits of a very general class of birth-death processes
- Contribution 2: For cyclic systems the scaling limits give stationary distribution approximations. This is a different novel application of scaling (fluid) limits (the "usual": Transient analysis, Stability, Near-Optimal control of queueing systems)
- Future work: Change point detection (and prediction) of birth-death processes using scaling limits

A General Formulation

- $X_N(\cdot)$ a sequence of processes with rates $\lambda_N(y), \mu_N(y)$
- $\mathcal{C}_N$, $N = 1, 2, \ldots$ a sequence of subsets of the state space
- $x(\cdot)$, solution of $\dot{x}(t) = b\big(x(t)\big) - d\big(x(t)\big)$, $x(0) = X(0)$

Notation: $\bar{g}(\mathcal{C}_N) = \sup_{y \in \mathcal{C}_N} g(y)$, for a function $g(\cdot)$

Theorem

Assume: (i) $\exists N_0 : \forall N \geq N_0$, $\lfloor N\, x(t) \rfloor \in \mathcal{C}_N$

(ii) $\bar{\lambda}_N(\mathcal{C}_N) = o(N^2)$, $\bar{\mu}_N(\mathcal{C}_N) = o(N^2)$

(iii) $\exists L, \forall N, \forall y \in \mathcal{C}_N, \forall y'$:

$$\left| \frac{\lambda_N(y)}{N} - b(y') \right| \leq L \left| \frac{y}{N} - y' \right|, \quad \left| \frac{\mu_N(y)}{N} - d(y') \right| \leq L \left| \frac{y}{N} - y' \right|$$

Then,

$$\lim_{N \to \infty} P\Big(\sup_{s \in [0,t]} \left| \frac{X_N(s)}{N} - x(s) \right| > \epsilon \Big) = 0$$

Similar result holds for hitting times