Non-Existence of Stabilising Policies for the Critical Push-Pull Network and Generalisations

Yoni Nazarathy[(1)], Leonardo Rojas-Nandayapa[(1)], Thomas S. Salisbury[(2)].

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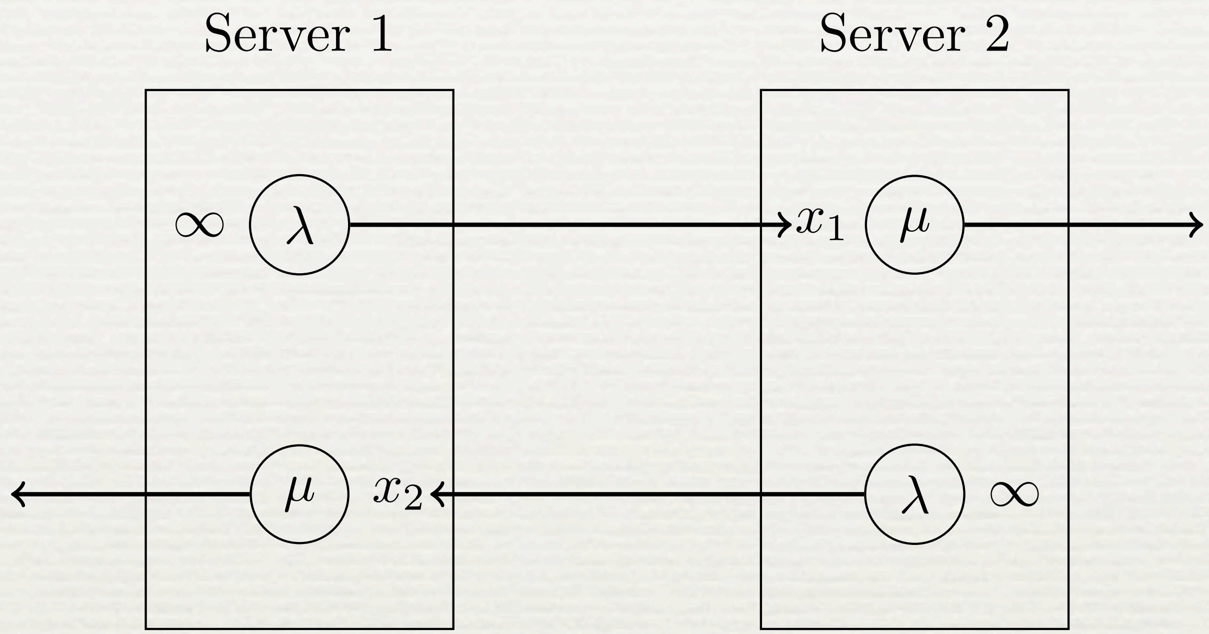
(1) The University of Queensland, Brisbane, Australia
(2) York University, Toronto, Canada

The Push-Pull Queueing Network

Kopzon - Weiss (2002), Kopzon - N - Weiss (2009), N - Weiss (2010), Lefeber - Weiss (...)

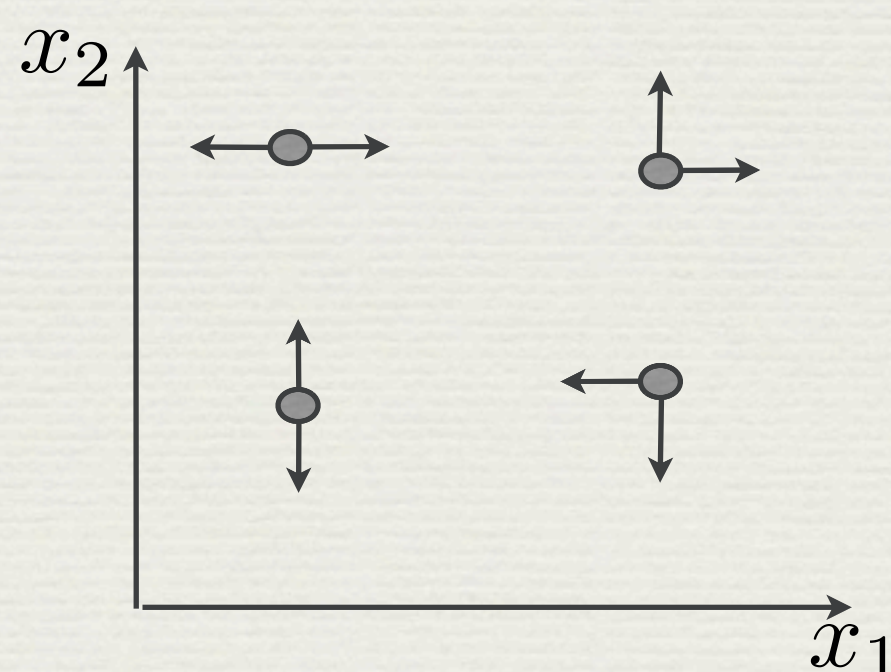


Look for a stationary, deterministic, preemptive, memory-less, **non-idling**, central control law (policy) : $\mathcal{P} : \mathbb{Z}_+^2 \to \{\text{push}, \text{pull}\}^2$

Each policy yields a Markov chain on $\mathbb{Z}_+^2$

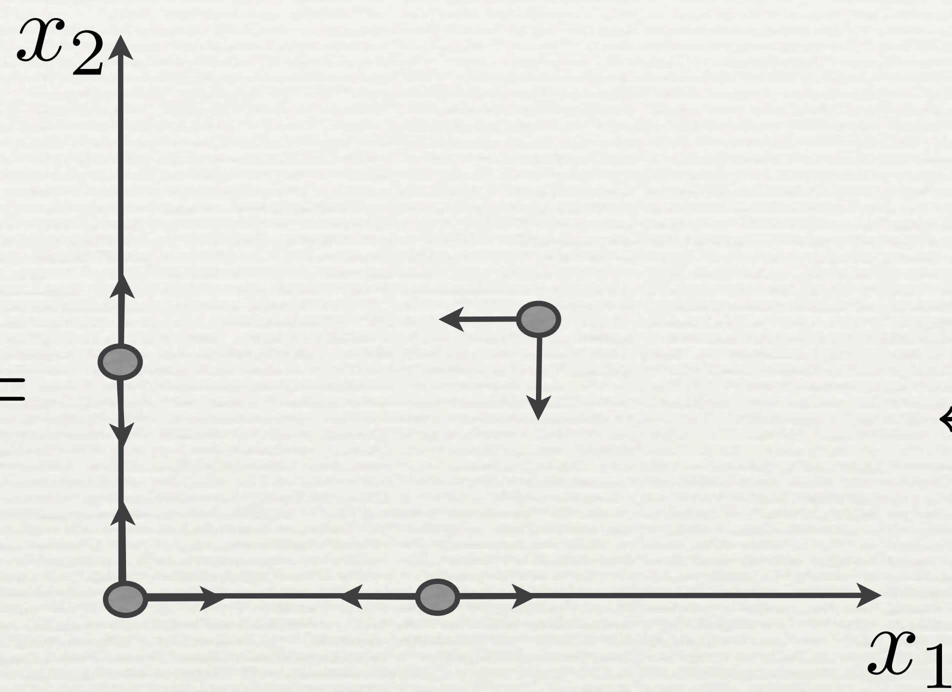
A policy is **stable** if the resulting Markov chain has a positive recurrent class, reached w.p. 1

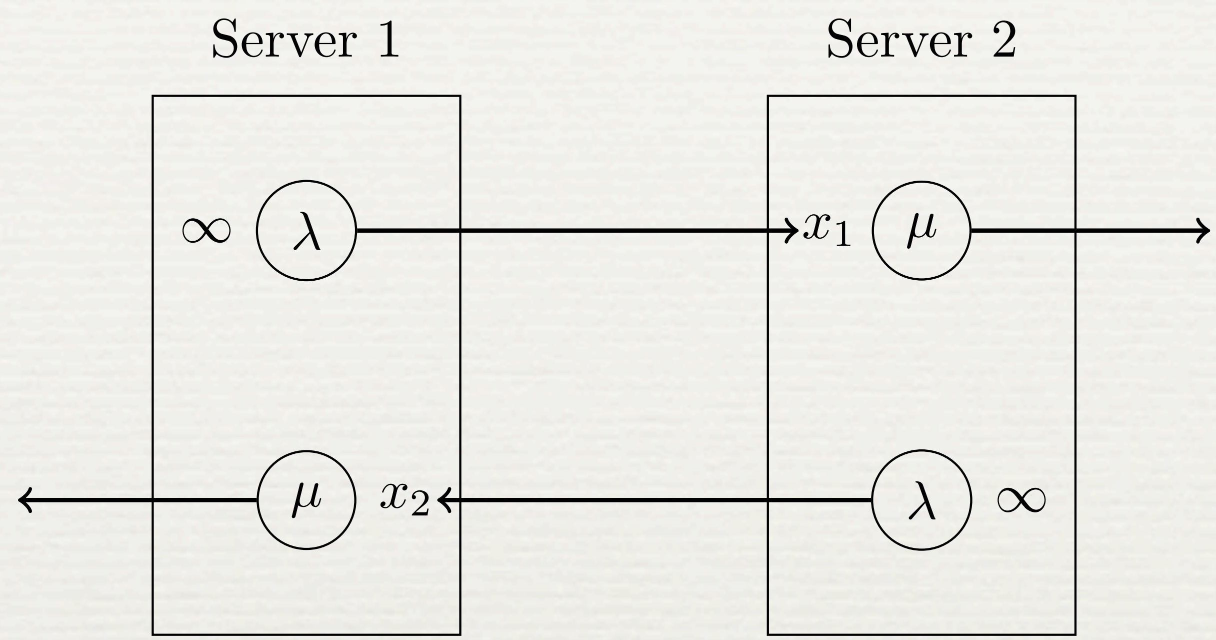
Does such a policy exist?



“Pull-Priority” Policy

$$\mathcal{P}\big((x_1, x_2)\big) =$$

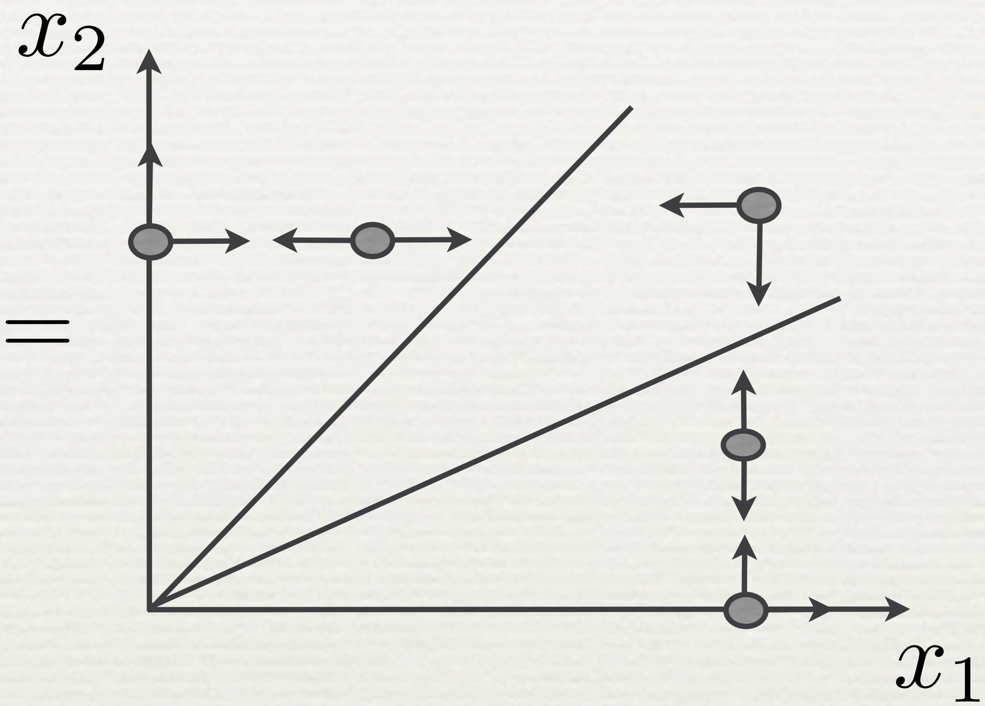


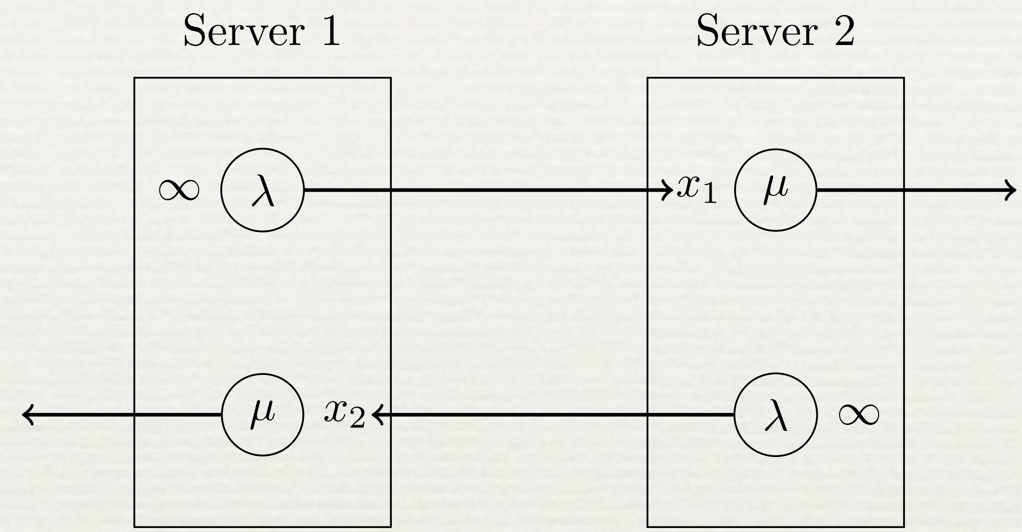


Stable iff $\lambda < \mu$

"Thresholds Policy"

$$\mathcal{P}\big((x_1, x_2)\big) =$$

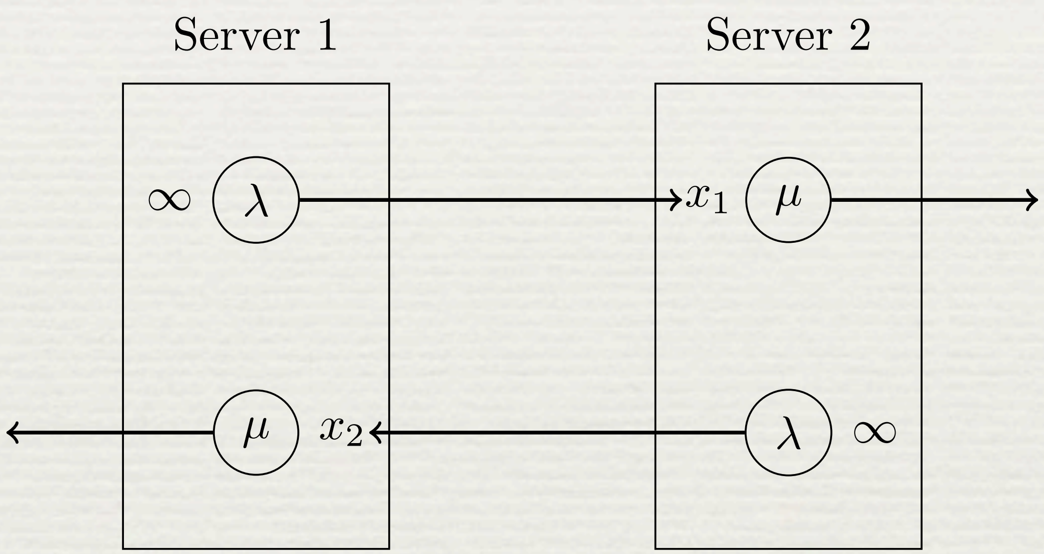


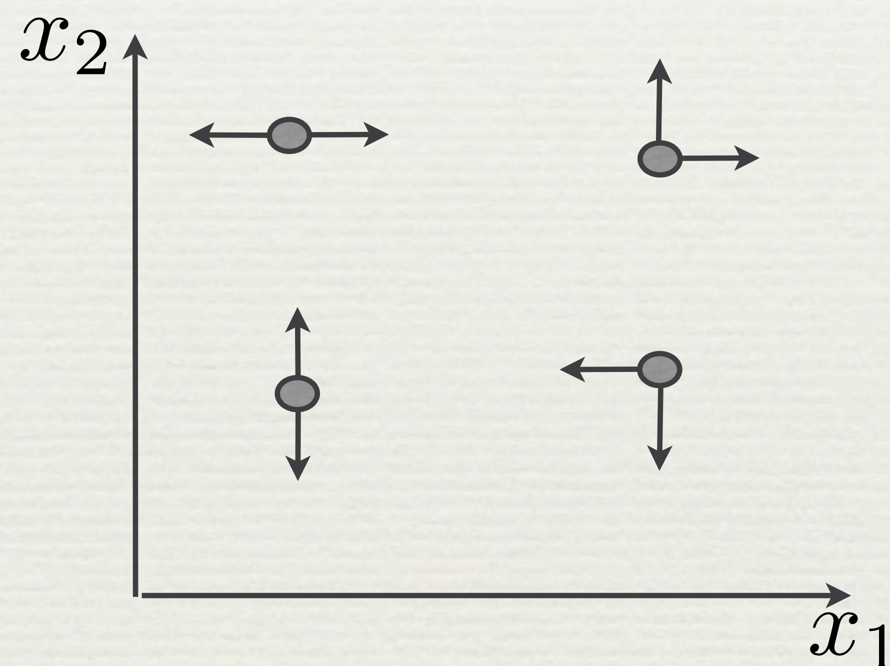


Stable iff $\lambda > \mu$

Intermediate Summary

Stable policies exist for both the $\lambda < \mu$ case and the $\lambda > \mu$ case.

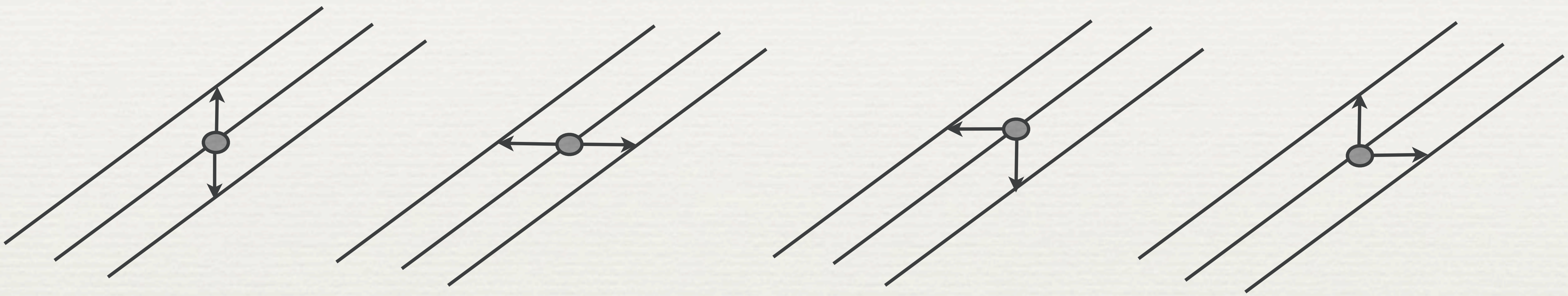




What about the critical $\lambda = \mu$ case?

Proposition: If $\lambda = \mu$ there does not exist a stable $\mathcal{P}$

Proof: $g((x_1, x_2)) = x_1 - x_2$, $Z_n = g(X_n)$ is a martingale for any $\mathcal{P}$



Assume $\exists$ positive recurrent $\mathcal{B} \subset \mathbb{Z}_+^2$. Take $\mathbf{x}, \mathbf{y} \in \mathcal{B}$ with $g(\mathbf{x}) \neq g(\mathbf{y})$

Set $X_0 = \mathbf{x}$ w.p. 1 and define $T = \inf\{n \geq 0 : X_n = \mathbf{y}\}$

For $\mathcal{B}$ positive recurrent, $\mathbb{E}\,[T] < \infty$, so:

$$g(\mathbf{x}) = \mathbb{E}[Z_0] = \mathbb{E}[Z_T] = g(\mathbf{y})$$

a contradiction.

Generalisation to "Homogenous" Controlled Random Walks

$X_n \in \mathbb{Z}_+^M$ with actions $\mathcal{A}(\mathbf{z})$ for state $\mathbf{z}$ yielding "limited" jumps

To show non-stabilisabily find $g(\cdot)$:

$$\mathbb{E}_a[g(X_{n+1}) - g(X_n) \,|\, X_n = \mathbf{z}] = 0, \quad \mathbf{z} \in \mathbb{Z}_+^M, \; a \in \mathcal{A}(\mathbf{z}).$$

Set $\mathcal{A} = \bigcup_{\mathbf{z}} \mathcal{A}(\mathbf{z})$, assume $|\mathcal{A}| < \infty$ and $\mathbb{P}_a(\cdot \,|\, \mathbf{z})$ "same" for all $\mathbf{z}$

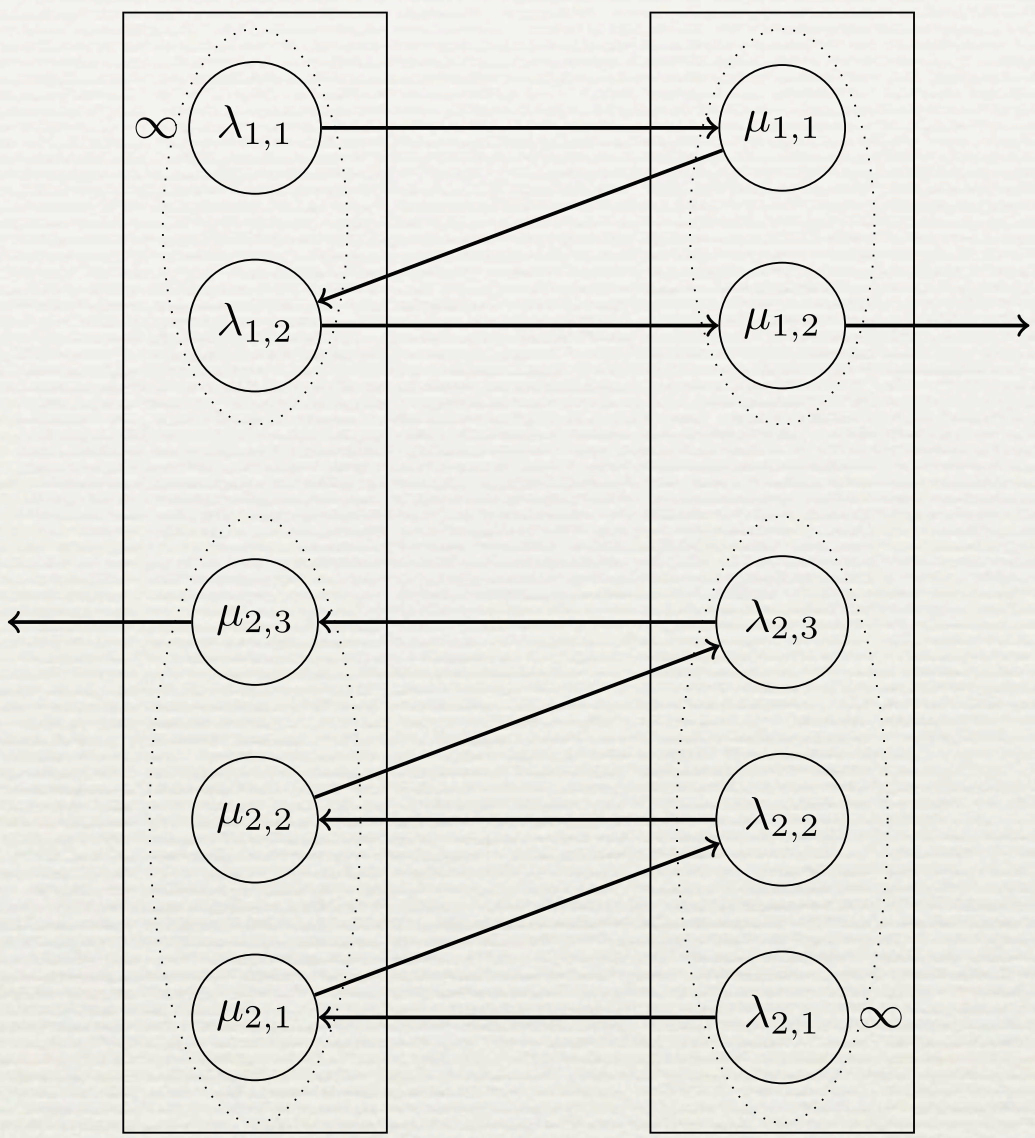
Then we have $|\mathcal{A}|$ equations for a harmonic function $g(\cdot)$:

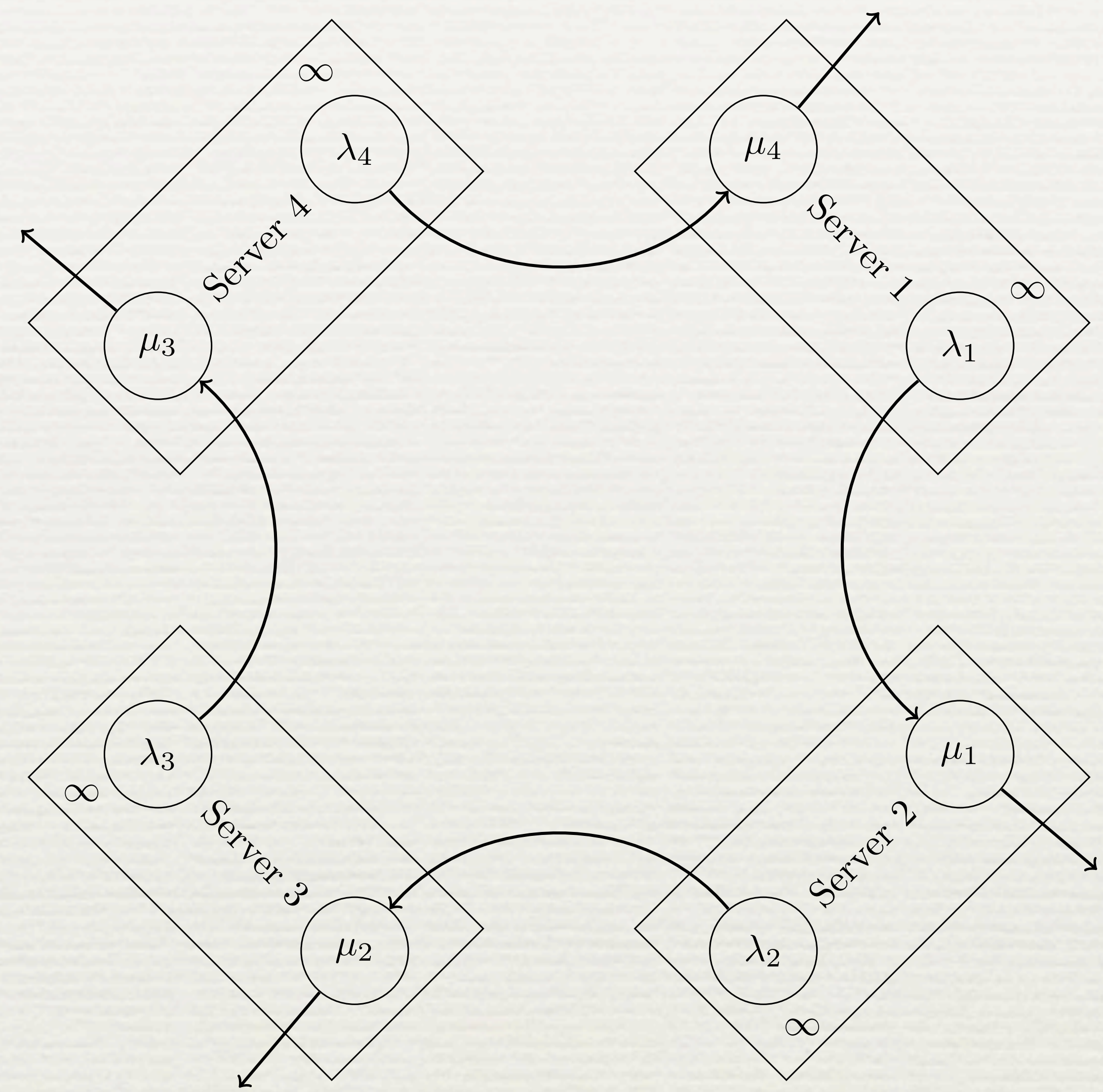
Go for linear $g(\cdot)$: $g(\mathbf{z}) = \boldsymbol{\alpha}'\mathbf{z}$

$$\boldsymbol{\alpha}' \, \mathbb{E}_a[X_{n+1} - X_n \,|\, X_n] = 0, \; a \in \mathcal{A} \quad \text{or} \quad \mathbf{D}\boldsymbol{\alpha} = \mathbf{0}$$

Proposition: The homogenous controlled random walk is non-stabilisable if $\text{rank}(\mathbf{D}) < M$.

Application to Generalisations of the Push-Pull





Critical: $\frac{1}{\sum_{j=1}^{N_i} \frac{1}{\lambda_{i,j}}} = \frac{1}{\sum_{j=1}^{M_i} \frac{1}{\mu_{i,j}}}, \; i = 1, 2$

Critical: $\lambda_i = \mu_i, \; i = 1, \ldots, M$

Structured matrixes:

$\mathbf{D}_{(N_1+M_2)(N_2+M_1)\times(N_1+N_2+M_1+M_2-2)}$

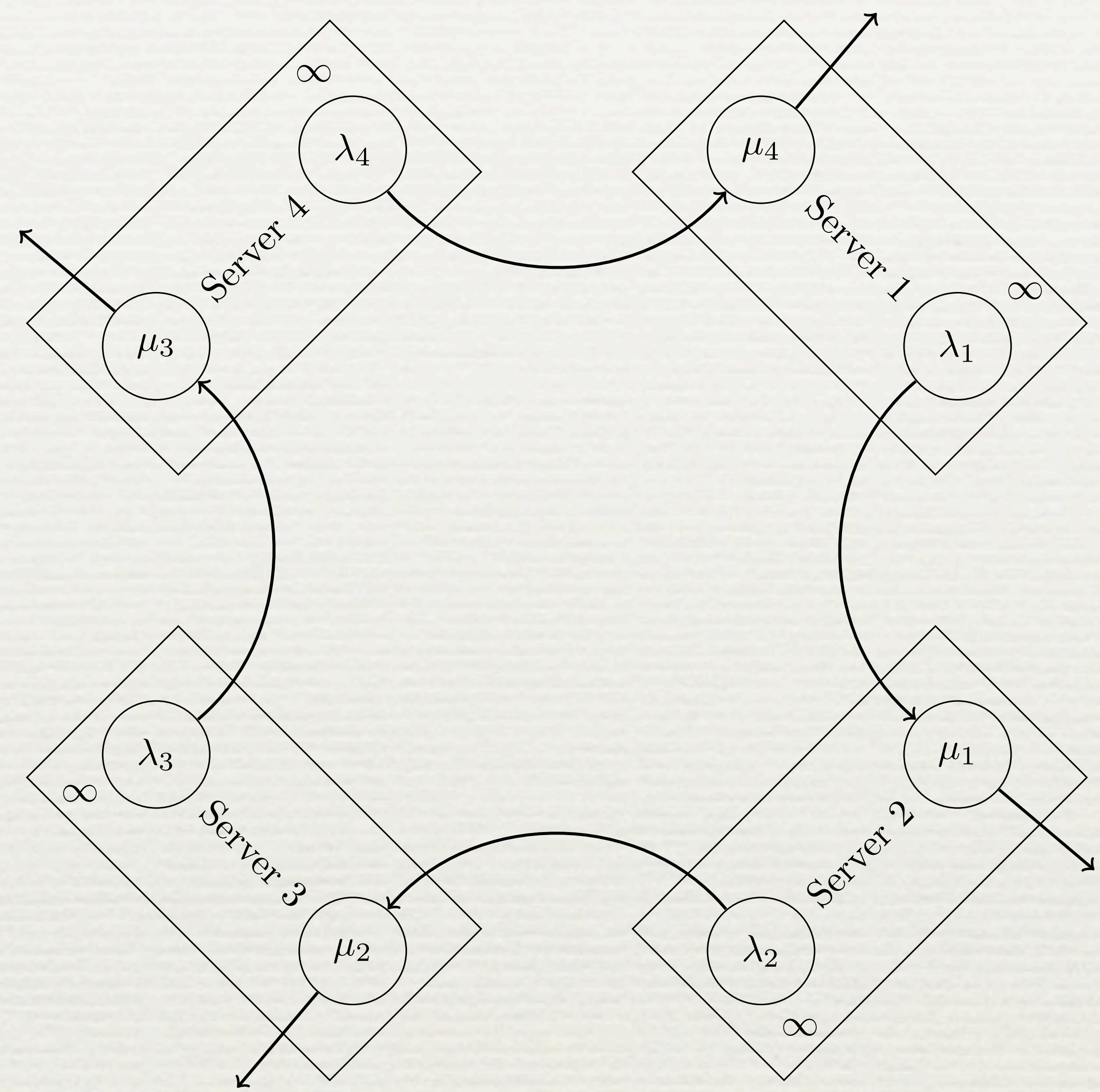
$\mathbf{D}_{2^M \times M}$

The Structure of $\mathbf{D}$

Action on Server $i-1$	Action on Server i	Drift of Queue i
push	push	λ_i
push	pull	$\lambda_i - \mu_i := \delta_i$
pull	push	0
pull	pull	$-\mu_i$

i'th entry	$i+1$'th entry
1	1 or δ
-1	-1 or 0
0	δ or 1
δ	0 or -1

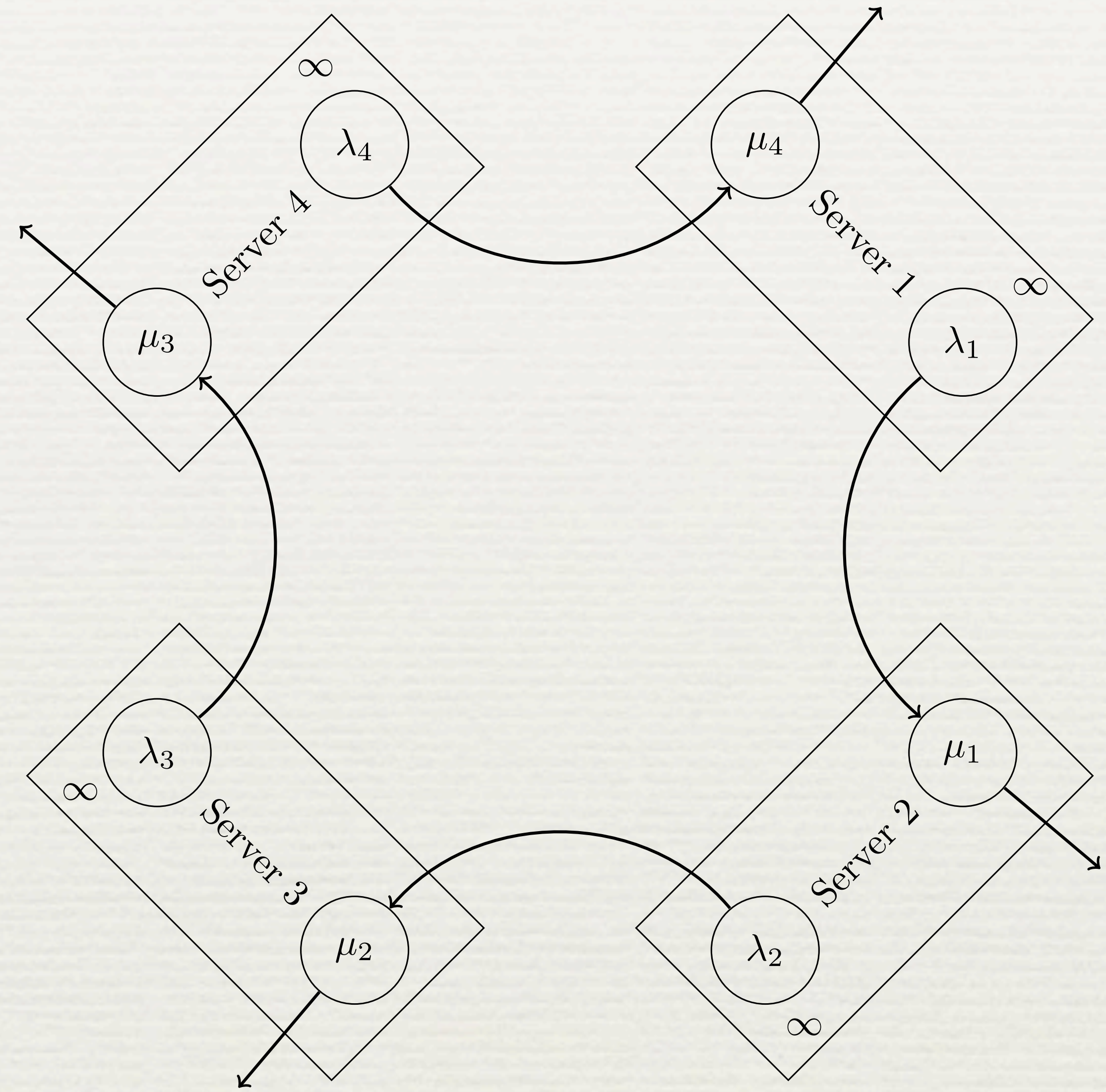
1. There are no adjacent -1's and 1's.
2. The number of 0's between **any** two entries with the same sign is zero or **even**.
3. The number of 0's between **any** two entries with opposite signs is **odd**.
4. The number of 0's is **even**.



Proposition: For critical rings

$$\mathrm{rank}(\mathbf{D}_{2^M \times M}) = \begin{cases} M-1 & \text{if } M \text{ is even,} \\ M & \text{if } M \text{ is odd.} \end{cases}$$

So critical rings with
M even are non-stabilisable



Critical rings with
M odd are stable with “pull-priority”

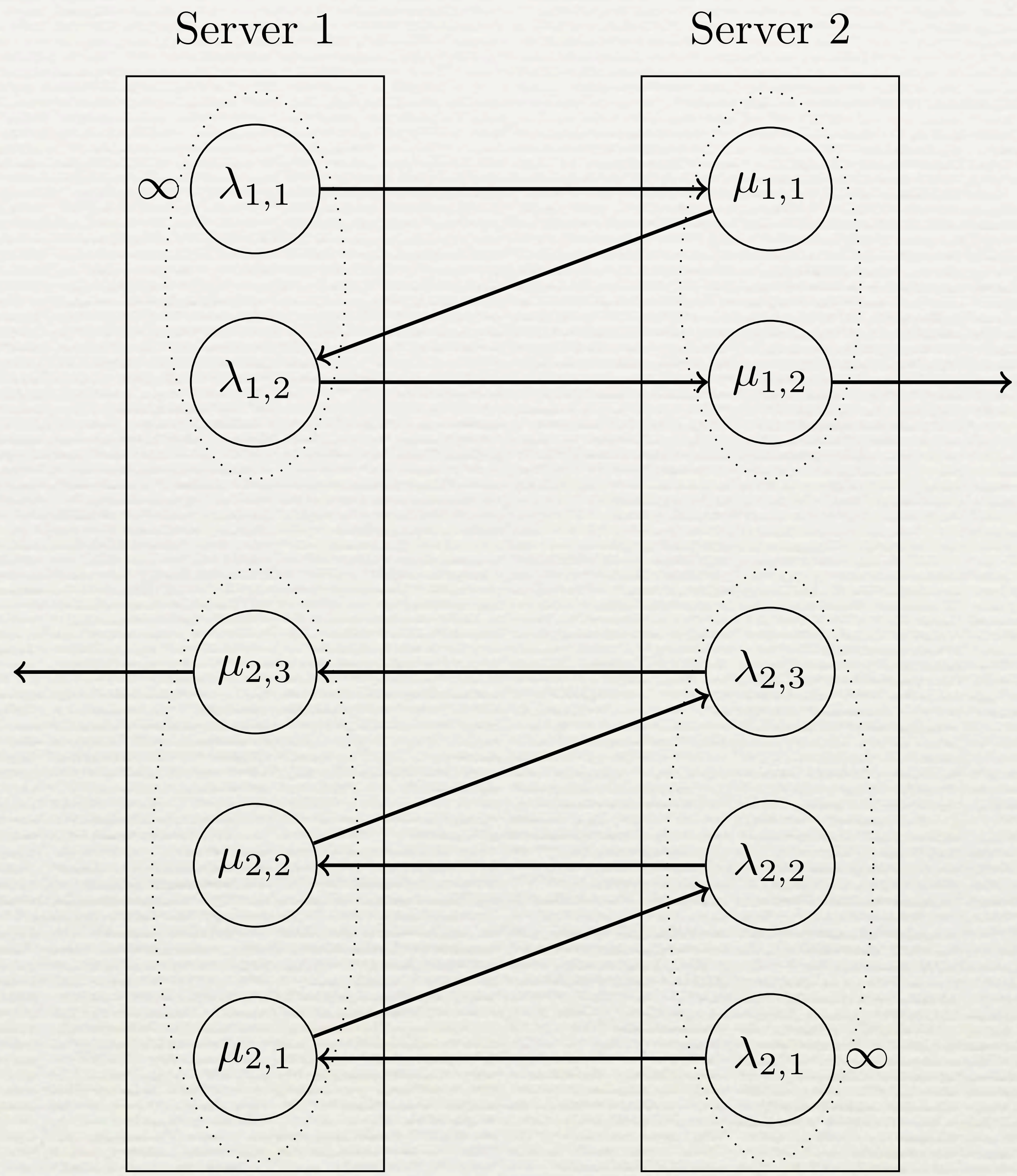
Guo - Lefeber - N - Zhang - Weiss (2011...)

$$\text{rank}\big(\mathbf{D}_{2^M \times M}\big) = \begin{cases} M-1 & \text{if } M \text{ is even,} \\ M & \text{if } M \text{ is odd.} \end{cases}$$

Re-entrant lines on Two Servers

Proposition:

If $\frac{1}{\sum_{j=1}^{N_i} \frac{1}{\lambda_{i,j}}} = \frac{1}{\sum_{j=1}^{M_i} \frac{1}{\mu_{i,j}}}, i = 1, 2$ then the network is non-stabilisable

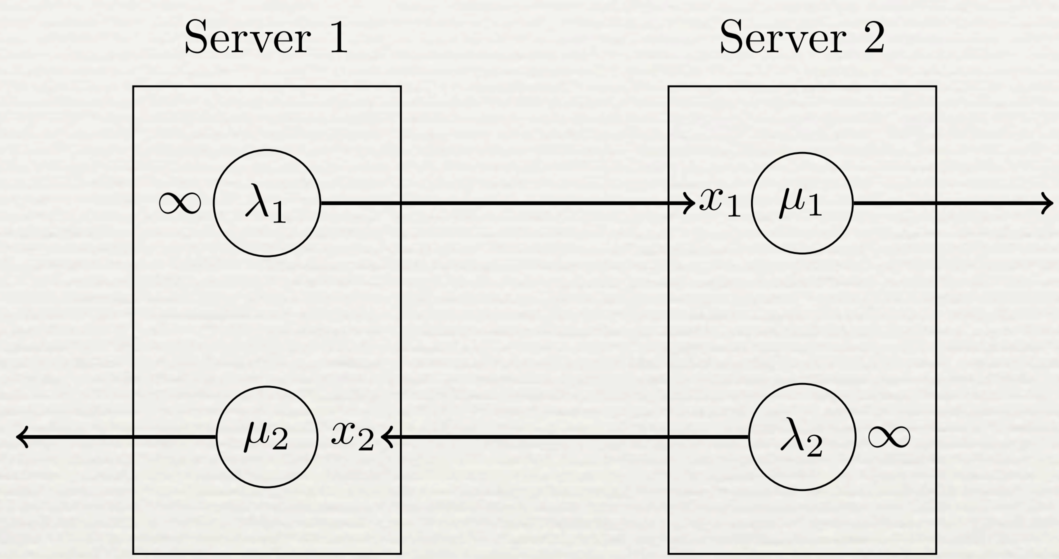


Note: If $\frac{1}{\sum_{j=1}^{N_i} \frac{1}{\lambda_{i,j}}} < \frac{1}{\sum_{j=1}^{M_i} \frac{1}{\mu_{i,j}}}, i = 1, 2$ then "pull-priority-LBFS" is stable

Guo - Lefeber - N - Zhang - Weiss (2011...)

In summary...
Stability analysis of networks is fun

Extra: Necessary conditions for rate stability



$$\theta_i = \lim_{t\to\infty} \frac{T_i(t)}{t}$$

$$\begin{aligned} \theta_1\lambda_1 &= (1-\theta_2)\mu_1 \\ (1-\theta_1)\mu_2 &= \theta_2\lambda_2 \end{aligned}$$

$$(\theta_1, \theta_2) = \left(\frac{\mu_1(\lambda_2-\mu_2)}{\lambda_1\lambda_2-\mu_1\mu_2},\ \frac{\mu_2(\lambda_1-\mu_1)}{\lambda_1\lambda_2-\mu_1\mu_2}\right)$$