Instability, Stability and Non-Stabilizability of Queueing Networks

Yoni Nazarathy,
The University of Queensland,

Based on some joint work with

Erjen Lefeber, Eindhoven University of Technology,
Leonardo Rojas-Nandayapa, The University of Queensland,
Tom Salisbury, York University,
Gideon Weiss, The University of Haifa and The University of Southern California,
Hanqin Zhang, National University of Singapore.

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Queues and Queueing Networks
Congestion, delay and resource scarcity occurs in a variety of application areas:

- Customer service systems
- Complex manufacturing lines
- Telecommunication networks and computing systems
- Transportation networks
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Stochastic queueing network models often capture the essentials of such examples allowing for quantitative performance evaluation, optimization and control.
The Study of Queueing Networks in Applied Probability and Operations Research

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Researchers “doing queueing” currently at UQ

Dirk Kroese, Ross McVinish, Y.N., Phil Pollett, Leonardo Rojas-Nandayapa,...
A Single Queue

Items arrive at random times to a server, queue up, each requiring service for a random duration, then depart
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Construction of the Queue Length Process: $Q(t)$

$A(t) \equiv$ counting process generated by a sequence of random \textbf{inter-arrival times} each with mean $\lambda^{-1}$

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\[
Q(t) = Q(0) + A(t) - S(T(t))
\]

\[
T(t) = \int_0^t 1_{\{Q(s) > 0\}} \, ds
\]

The Load \(\rho = \frac{\lambda}{\mu}\)
- \(\rho < 1\) queue is stable
- \(\rho > 1\) queue is unstable
- \(\rho = 1\) queue is critically unstable
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The Flavor of Classic Queueing Network Results

\[ \lambda_1 = \alpha_1 + p_{2,1} \]

\[ \lambda_2 = \alpha_2 + p_{1,2} \]

\[ \rho_i = \frac{\lambda_i}{\mu_i}, \quad i = 1, 2. \]

A Product Form Result

Assume Poisson arrival and service processes. If \( \rho_1, \rho_2 < 1 \) then,

\[ \lim_{t \to \infty} P(Q_1(t) = k_1, Q_2(t) = k_2) = \prod_{i=1}^{2} (1 - \rho_i) \rho_{k_i} \]

otherwise the network is not stable.

Note: Without the Poisson assumptions the product form typically does not hold, yet the stability properties are the same.
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Control Policies

Now there is a choice as to how to allocate server resources:

**Policy:** What operation should be served by each of the servers at every time instant based on the current state
Multi-Class Queueing Networks

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*A realistic research goal: understanding stability*
Instability
The Kumar-Seidman-Rybko-Stolyar Network

Load Conditions

Necessary condition for stability:

\[ \rho_1 = \alpha_1 \frac{1}{\lambda_1} + \alpha_2 \frac{1}{\mu_2} < 1, \quad \rho_2 = \alpha_1 \frac{1}{\mu_1} + \alpha_2 \frac{1}{\lambda_2} < 1. \]

A Control Question

If \( \rho_i < 1, \ i = 1, 2 \), are all work conserving policies stabilizing?

KSRS Adversarial Idea: Try the pull-priority Policy

Give priority to pull operations, \( \mu_1, \mu_2 \), over push operations \( \lambda_1, \lambda_2 \).
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**KSRS Adversarial Idea:** Try the **pull-priority** Policy

Give priority to pull operations, \( \mu_1, \mu_2, \) over push operations \( \lambda_1, \lambda_2 \)
Illustrative example in which the load conditions hold:

$$(\alpha_i = 3, \lambda_i = 10, \mu_i = 5) \implies \rho_i = 9/10, \quad i = 1, 2.$$
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Illustration of instability by means of deterministic **fluid** dynamics:
Dynamics of Pull-Priority KSRS

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Illustration of instability by means of deterministic \textbf{fluid} dynamics:

\[
\begin{align*}
\alpha - \lambda &= -7 & \lambda - \mu &= +5 \\
0 &= \alpha &= +3 & 0 &= \mu &= +5 \\
\alpha - \alpha &= 0 & \alpha - \mu &= -2 \\
0 &= \alpha &= +3 & 0 &= \lambda - \mu &= -7 \\
\alpha + 0 &= \alpha &= +3 & 0 &= \alpha - \lambda &= -2 \\
\alpha + 0 &= \alpha &= +3 & 0 &= \alpha - \alpha &= 0 \\
0 &= \alpha &= +3 & 0 &= \lambda - \mu &= +5
\end{align*}
\]

A Virtual Server

Observation: Pull operations “never” occur at the same time. Thus with this policy, an additional condition for stability is:

\[
\rho_v := \alpha_1 \frac{1}{\mu_1} + \alpha_2 \frac{1}{\mu_2} < 1
\]
Lessons Learned from KSRS

- Stability is not just a property of the network but rather of both the network and the control policy – this is in stark difference to classic queueing networks.
- “Innocent looking” control policies can be very bad.
- This is easy to detect (and fix) for small toy examples such as KSRS - but what about big complex manufacturing networks?
- The sub-field of stability analysis of queueing networks was “born” (early 90’s).

Summarizing Books (including KSRS and beyond)

Queueing Networks with Infinite Supplies
A Different Kind of Model

Many real life systems often operate some servers at full utilization yet in previous models $\rho = 1$ implies critically unstable behavior.
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### Infinite Supply Models

- Assume that servers generate arrivals to the network by having some of the queues that never run out of work.
- This allows full utilization of servers.
- Analyze stability for non-idling (fully-utilizing) control policies.
A Different Kind of Model

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Infinite Supply Models

- Assume that servers generate arrivals to the network by having some of the queues that never run out of work
- This allows full utilization of servers
- Analyze stability for non-idling (fully-utilizing) control policies

The simplest example is Gideon Weiss’s push-pull network:
The Push-Pull Queueing Network as a Markov Chain

Under Poisson assumptions, every control policy $P: \mathbb{Z}^2_+ \rightarrow \{\text{push}, \text{pull}\}$ (with restrictions at the axes) implies a Markov chain on $\mathbb{Z}^2_+$. 

Server 1

$\infty \quad \lambda_1 \quad Q_1 \quad \mu_2 \quad Q_2$

Server 2

$\mu_1 \quad Q_1 \quad \lambda_2 \quad \infty$
Policies and Markov Chains

Under **Poisson assumptions**, every control policy $\mathcal{P} : \mathbb{Z}_+^2 \rightarrow \{\text{push, pull}\}^2$ (with restrictions at the axes) implies a Markov chain on $\mathbb{Z}_+^2$. 
The Push-Pull Network with $\lambda_i < \mu_i$
The Push-Pull Network with $\lambda_i < \mu_i$

Pull-Priority Policy is Stabilizing
The Push-Pull Network with $\lambda_i > \mu_i$
The Push-Pull Network with $\lambda_i > \mu_i$

A Threshold Policy is Stabilizing
Is there a $P : \mathbb{Z}_+^2 \to \{\text{push, pull}\}^2$ (with restrictions at the axes) such that the Markov chain has a **positive recurrent** state that is reached w.p. 1?
Is there a $\mathcal{P} : \mathbb{Z}_+^2 \to \{\text{push, pull}\}^2$ (with restrictions at the axes) such that the Markov chain has a positive recurrent state that is reached w.p. 1?

We’ll get back to this in a few minutes....
Push-Pull Rings
Generalizes the Push to $M$ servers

- $M$ queues, each with "potential load", $\gamma_i = \frac{\lambda_i}{\mu_i}$
- Pull-priority policy
- An interesting case is when $\gamma_i > 1$ but "not too large": It turns out that odd rings are stable yet even rings are not
Stochastic Model and Fluid Model

**Stochastic Model \((Q(t), T(t))\)**

\[
\begin{align*}
Q_i(t) &= Q_i(0) + S_{i,1}(T_{i,1}(t)) - S_{i,2}(T_{i,2}(t)) \\
t &= T_{i,1}(t) + T_{i-1,2}(t) \\
0 &= \int_0^t Q_i(s) dT_{i+1,1}(s)
\end{align*}
\]

**Associated Fluid Model \((\bar{Q}(t), \bar{T}(t))\)**

\[
\begin{align*}
\bar{Q}_i(t) &= \bar{Q}_i(0) + \lambda_i \bar{T}_{i,1}(t) - \mu_i \bar{T}_{i,2}(t) \\
t &= \bar{T}_{i,1}(t) + \bar{T}_{i-1,2}(t) \\
0 &= \int_0^t \bar{Q}_i(s) d\bar{T}_{i+1,1}(s)
\end{align*}
\]
Thm: (Dai ’95), adapted to infinite supplies

Assume minor technical assumptions on the processing time distributions. If there exists a $\tau$ such that for all solutions of the fluid model and all $t \geq \tau$, $\sum Q_i(t) = 0$ then the (stochastic) network is stable.

- All solutions of the fluid model are Lipschitz, thus have derivatives a.e.
- **Regular time points**: Time points at which derivatives exists

Lemma

If we have a Lyapounov function: $V : \mathbb{R}^M \rightarrow \mathbb{R}$ such that for all regular time points of all solutions of the fluid model, $\frac{d}{dt} V(\bar{Q}(t)) < -\epsilon$ for some $\epsilon > 0$, then the fluid model is stable.
Stability Result: $M$ odd, $\gamma_i > 1$

Theorem (Erjen Lefeber, Gideon Weiss, Y.N.)

The push-pull ring with $M$ odd, $\gamma_i > 1$ for all $i$, operating under a pull-priority policy is stable if $\Delta < 0$, where

$$\Delta = \sum_{i=1}^{M} c_i \left( \frac{M-1}{2} (\gamma_i - 1) - 1 \right),$$

with,

$$c_i = (((\gamma_{i-1} - 1)\gamma_{i-2} + 1)\gamma_{i-3} - 1)\gamma_{i-4} \cdots \cdots \cdots \gamma_{i+2} - 1)\gamma_{i+1} + 1.$$ 

Note: If $\gamma_i = \gamma$ for all $i$ then the stability condition reduces to:

$$\gamma < 1 + \frac{1}{M-1}$$
1. Use $V(x) = \sum_{i=1}^{M} c_i x_i$ as Lyapunov function for the fluid model with coefficients, $c_i$, designed based on the intuition that “typical” fluid trajectories eventually cycle on states of the form (e.g. $M = 5$):

$(+, 0, +, 0, +), (+, +, 0, +, 0), (0, +, +, 0, +), (+, 0, +, +, 0), (0, +, 0, +, +)$.

The $c_i$ are such that $\dot{V}(t)$ is constant during such cycles:

$$
\begin{bmatrix}
-1 & 0 & \gamma_3 - 1 & 0 & \gamma_5 - 1 \\
\gamma_1 - 1 & -1 & 0 & \gamma_4 - 1 & 0 \\
0 & \gamma_2 - 1 & -1 & 0 & \gamma_5 - 1 \\
\gamma_1 - 1 & 0 & \gamma_3 - 1 & -1 & 0 \\
0 & \gamma_2 - 1 & 0 & \gamma_4 - 1 & -1
\end{bmatrix}
\begin{bmatrix}
c_1 \\
c_2 \\
c_3 \\
c_4 \\
c_5
\end{bmatrix}
= \begin{bmatrix}
\Delta \\
\Delta \\
\Delta \\
\Delta \\
\Delta
\end{bmatrix}
$$

2. Characterize the regular time points and show that on these time points the Lyapunov function has negative drift.

Non-Stabilizability
Is there a $P : \mathbb{Z}^2_+ \to \{\text{push, pull}\}^2$ (with restrictions at the axes) such that the Markov chain has a positive recurrent state that is reached w.p. 1?
Is there a $\mathcal{P} : \mathbb{Z}_+^2 \to \{\text{push}, \text{pull}\}^2$ (with restrictions at the axes) such that the Markov chain has a positive recurrent state that is reached w.p. 1?

Theorem (Leonardo Rojas-Nandayapa, Tom Salisbury, Y.N.)

The push-pull network with $\lambda_i = \mu_i$, $i = 1, 2$ is non-stabilizable.
Non-stabilizability Proof

(This version assumes: \( \lambda_1 = \mu_1 = \mu_2 = \lambda_2 \) for simplicity)

1. Set \( x_n \) as the embedded Markov chain of \( X(t) = (Q_1(t), Q_2(t)) \)
2. Define \( g((x_1, x_2)) = x_1 - x_2 \) and \( Z_n = g(X_n) \)
   \( Z_n \) is a martingale for any \( \mathcal{P} \):

3. Assume \( \exists \) positive recurrent \( B \subset \mathbb{Z}_+^2 \).
   Take \( x, y \in B \) with \( g(x) \neq g(y) \)
4. Set \( X_0 = x \) and define \( T = \inf\{n \geq 0 : X_n = y\} \)
5. For \( B \) to be positive recurrent, \( E[T] < \infty \) so:
   \[ g(x) = E[Z_0] = E[Z_T] = g(y), \]
   a contradiction.
The idea of finding a linear-martingale simultaneously for all possible policies turns out to be fruitful in greater generality:
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**Theorem (Leonardo Rojas-Nandayapa, Tom Salisbury, Y.N.)**

Consider controlled queueing networks \( \{X_n, \ n \geq 0\} \) on \( \mathbb{Z}_+^M \) with \( L < \infty \) possible actions. Denote, by \( D \) the \( L \times M \) matrix with rows,

\[
\Delta_i := E_{\text{action } i}[X_{n+1} - X_n \mid X_n], \quad i = 1, \ldots, L.
\]

Then subject to technical non-degeneracy conditions, if

\[
\text{rank}(D) < M,
\]

then the network is non-stabilizable.
Corollary (Leonardo Rojas-Nandayapa, Tom Salisbury, Y.N.)

Push-Pull Rings with $M$ even and $\lambda_i = \mu_i$ are non-stabilizable.
Corollary (Leonardo Rojas-Nandayapa, Tom Salisbury, Y.N.)

Consider an infinite supply network with 2 servers and \( S \) streams. Assume,
\[
\sum_{j \in C_1(i)} \mu_{i,j}^{-1} = \sum_{j \in C_2(i)} \mu_{i,j}^{-1}, \quad i = 1, \ldots, S,
\]
then the network is non-stabilizable.
Wrap-up
Current Projects Related to Stability Properties of Queueing Networks

- With Leonardo Rojas-Nandayapa and Tom Salisbury: Non-stabilizability of similar models under general processing time assumptions (can not use Martingale method as is)

- With Erjen Lefeber and Dieter Armbruster: It is known that in certain cases stability depends on the distributional assumptions. We have an illustration of this phenomenon based on deterministic dynamical (hybrid) systems

- With Gideon Weiss and Erjen Lefeber: General stability results for general queueing networks with infinite supplies

- Long term interest: Designing and understanding stabilizing adaptive control methods for complex queueing networks