Instability, Stability and Non-Stabilizability of Queueing Networks

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The University of Queensland,

Based on some joint work with

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Leonardo Rojas-Nandayapa, The University of Queensland,
Tom Salisbury, York University,
Gideon Weiss, The University of Haifa and The University of Southern California,
Hanqin Zhang, National University of Singapore.

University of Queensland, Mathematics Colloquium,
October 8, 2012.

Contents

Queues and Queueing Networks

The Study of Queueing Networks in Applied Probability and Operations Research

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- Customer service systems
- Complex manufacturing lines
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Researchers "doing queueing" currently at UQ

Dirk Kroese, Ross McVinish, Y.N., Phil Pollett,
Leonardo Rojas-Nandayapa,...

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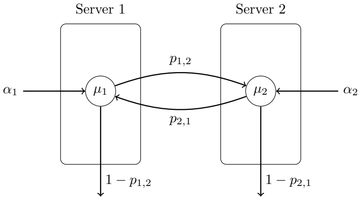
- $\rho = 1$ queue is **critically unstable**

The Flavor of Classic Queueing Network Results



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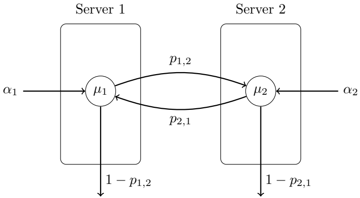

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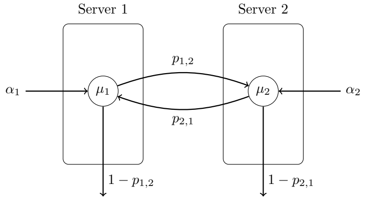

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A Product Form Result

Assume Poisson arrival and service processes. If $\rho_1, \rho_2 < 1$ then,

$$\lim_{t\to\infty} P\Big(Q_1(t) = k_1,\ Q_2(t) = k_2\Big) = \prod_{i=1}^{2}(1-\rho_i)\rho_i^{k_i},$$

otherwise the network is not stable.

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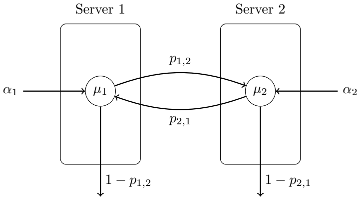

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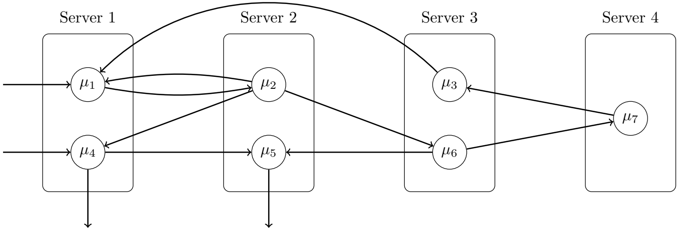
Note: Without the Poisson assumptions the product form typically does not hold, yet the stability properties are the same

Multi-Class Queueing Networks



Control Policies

Now there is a choice as to how to allocate server resources:

Policy: What operation should be served by each of the servers at every time instant based on the current state

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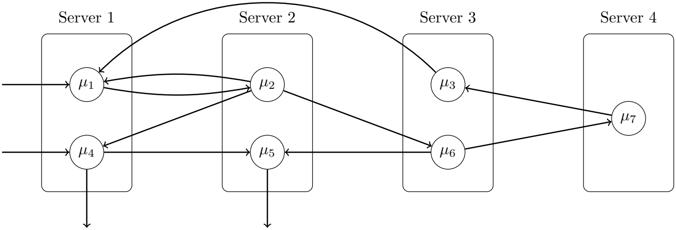

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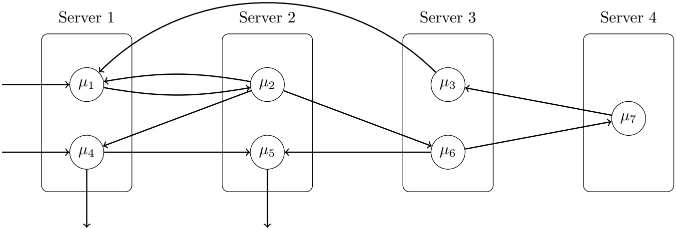

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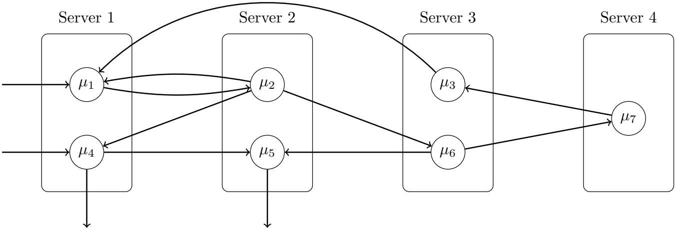

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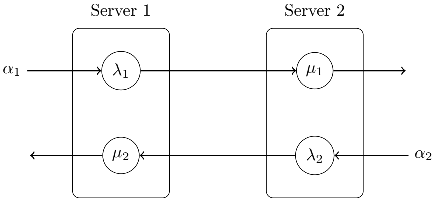
A realistic research goal: understanding stability

Instability

The Kumar-Seidman-Rybko-Stolyar Network



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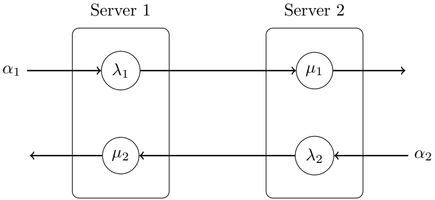

Load Conditions

Necessary condition for stability:

$$\rho_1 = \alpha_1 \frac{1}{\lambda_1} + \alpha_2 \frac{1}{\mu_2} < 1, \qquad \rho_2 = \alpha_1 \frac{1}{\mu_1} + \alpha_2 \frac{1}{\lambda_2} < 1.$$

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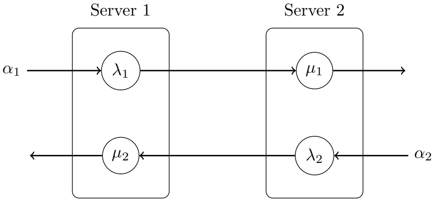

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If $\rho_i < 1$, $i = 1, 2$, are all **work conserving** policies stabilizing?

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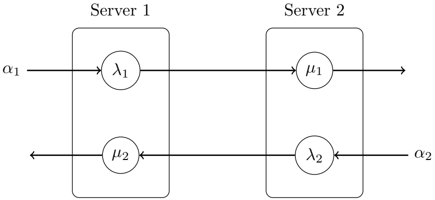

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KSRS Adversarial Idea: Try the **pull-priority** Policy

Give priority to pull operations, μ_1, μ_2, over push operations λ_1, λ_2

Dynamics of Pull-Priority KSRS

Illustrative example in which the load conditions hold:

$$\left(\alpha_i = 3,\ \lambda_i = 10,\ \mu_i = 5\right) \quad \Longrightarrow \quad \rho_i = 9/10, \quad i = 1, 2.$$

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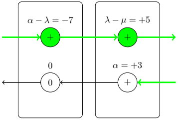

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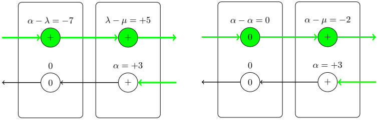

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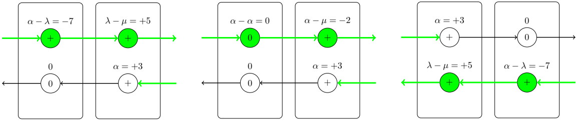

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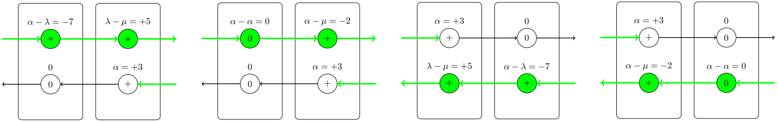

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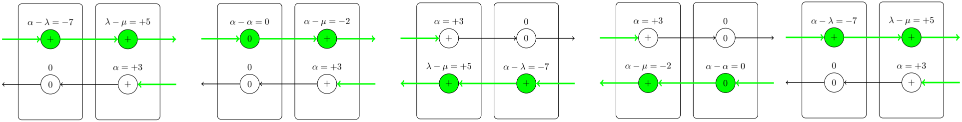

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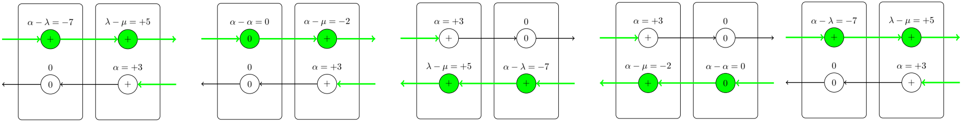

A Virtual Server

Observation: Pull operations "never" occur at the same time.
Thus with this policy, an additional condition for stability is:

$$\rho_v := \alpha_1 \frac{1}{\mu_1} + \alpha_2 \frac{1}{\mu_2} < 1$$

Some Notes and Comments

Lessons Learned from KSRS

- Stability is not just a property of the network but rather of both the network and the control policy – this is in stark difference to classic queueing networks
- "Innocent looking" control policies can be very bad
- This is easy to detect (and fix) for small toy examples such as KSRS - but what about big complex manufacturing networks?
- The sub-field of *stability analysis of queueing networks* was "born" (early 90's)

Summarizing Books (including KSRS and beyond)

- *Stability of Queueing Networks*, Maury Bramson, 2008.
- *Control Techniques for Complex Networks*, Sean Meyn, 2008.
- *Fundamentals of Queueing*, Hong Chen, David Yao, 2001.

Queueing Networks with Infinite Supplies

A Different Kind of Model

Many real life systems often operate some servers at full utilization yet in previous models $\rho = 1$ implies critically unstable behavior

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Infinite Supply Models

- Assume that severs generate arrivals to the network by having some of the queues that never run out of work
- This allows full utilization of servers
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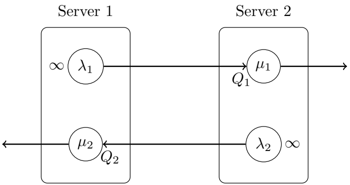
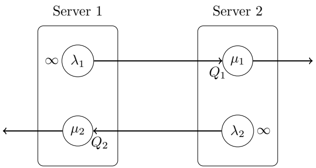
The simplest example is Gideon Weiss's **push-pull network**:



The Push-Pull Queueing Network as a Markov Chain



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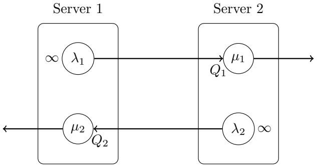
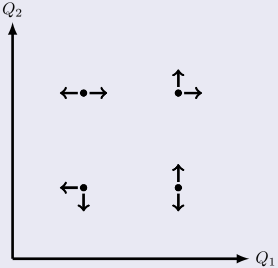
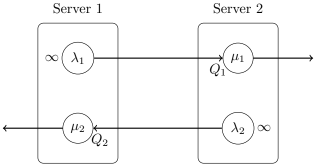


Policies and Markov Chains

Under **Poisson assumptions**, every control policy $\mathcal{P} : \mathbb{Z}_+^2 \to \{\text{push}, \text{pull}\}^2$ (with restrictions at the axes) implies a Markov chain on $\mathbb{Z}_+^2$



The Push-Pull Network with $\lambda_i < \mu_i$



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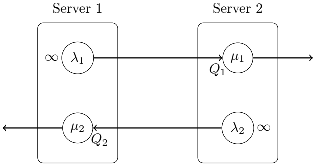
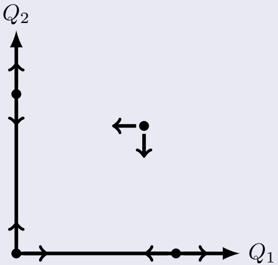


Pull-Priority Policy is Stabilizing



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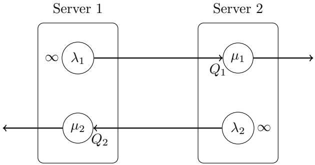

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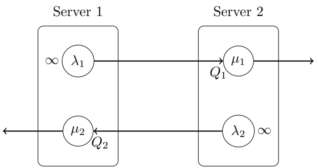
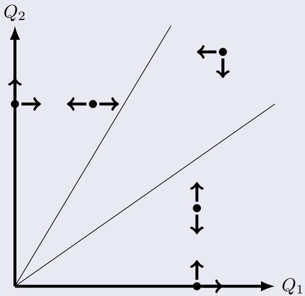
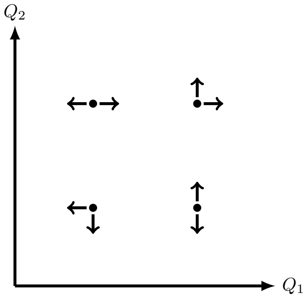


A Threshold Policy is Stabilizing



The Push-Pull Network with $\lambda_i = \mu_i$



Is there a $\mathcal{P} : \mathbb{Z}_+^2 \to \{\text{push}, \text{pull}\}^2$ (with restrictions at the axes) such that the Markov chain has a **positive recurrent** state that is reached w.p. 1?

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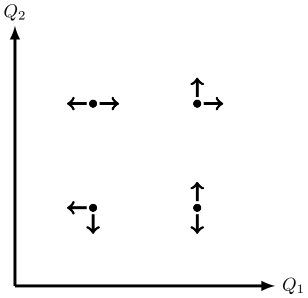

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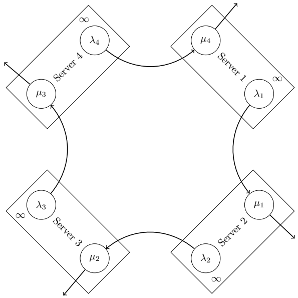
We'll get back to this in a few minutes....

Push-Pull Rings

Push-Pull Rings with Pull-Priority Policy



Generalizes the Push to M servers

- M queues, each with "potential load", $\gamma_i = \frac{\lambda_i}{\mu_i}$
- Pull-priority policy
- An interesting case is when $\gamma_i > 1$ but "not too large":
  It turns out that odd rings are stable yet even rings are not

Stochastic Model and Fluid Model

Stochastic Model $(Q(t), T(t))$

$$
\begin{aligned}
Q_i(t) &= Q_i(0) + S_{i,1}\big(T_{i,1}(t)\big) - S_{i,2}\big(T_{i,2}(t)\big) \\
t &= T_{i,1}(t) + T_{i-1,2}(t) \\
0 &= \int_0^t Q_i(s)\,dT_{i+1,1}(s)
\end{aligned}
$$

Associated Fluid Model $(\bar{Q}(t), \bar{T}(t))$

$$
\begin{aligned}
\bar{Q}_i(t) &= \bar{Q}_i(0) + \lambda_i \bar{T}_{i,1}(t) - \mu_i \bar{T}_{i,2}(t) \\
t &= \bar{T}_{i,1}(t) + \bar{T}_{i-1,2}(t) \\
0 &= \int_0^t \bar{Q}_i(s)\,d\bar{T}_{i+1,1}(s)
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Fluid Stability Framework

Thm: (Dai '95), adapted to infinite supplies

Assume minor technical assumptions on the processing time distributions. If there exists a τ such that for all solutions of the fluid model and all $t \geq \tau$, $\sum \bar{Q}_i(t) = 0$ then the (stochastic) network is stable.

- All solutions of the fluid model are Lipschitz, thus have derivatives a.e.
- **Regular time points**: Time points at which derivatives exists

Lemma

If we have a Lyapounov function: $V : \mathbb{R}^M \to \mathbb{R}$ such that for all regular time points of all solutions of the fluid model, $\frac{d}{dt} V\left(\bar{Q}(t)\right) < -\epsilon$ for some $\epsilon > 0$, then the fluid model is stable.

Stability Result: M odd, $\gamma_i > 1$

Theorem (Erjen Lefeber, Gideon Weiss, Y.N.)

The push-pull ring with M **odd**, $\gamma_i > 1$ for all i, operating under a pull-priority policy is stable if $\Delta < 0$, where

$$\Delta = \sum_{i=1}^{M} c_i \Big(\frac{M-1}{2}(\gamma_i - 1) - 1 \Big),$$

with,

$$c_i = ((\cdots(((\gamma_{i-1} - 1)\gamma_{i-2} + 1)\gamma_{i-3} - 1)\gamma_{i-4} \cdots\cdots\cdots)\gamma_{i+2} - 1)\gamma_{i+1} + 1.$$

Note: If $\gamma_i = \gamma$ for all i then the stability condition reduces to:

$$\gamma < 1 + \frac{1}{\frac{M-1}{2}}$$

Proof Overview

1. Use $V(x) = \sum_{i=1}^{M} c_i\, x_i$ as Lyapounov function for the fluid model with coefficients, c_i, designed based on the intuition that "typical" fluid trajectories eventually cycle on states of the form (e.g. $M = 5$):

   $$(+,0,+,0,+),\ \ (+,+,0,+,0),\ \ (0,+,+,0,+),\ \ (+,0,+,+,0),\ \ (0,+,0,+,+).$$

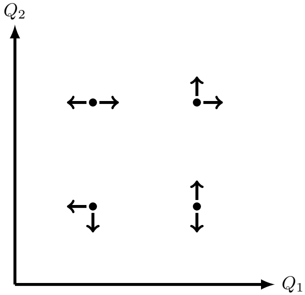
   The c_i are such that $\dot{V}(t)$ is constant during such cycles:

   $$\begin{bmatrix} -1 & 0 & \gamma_3-1 & 0 & \gamma_5-1 \\ \gamma_1-1 & -1 & 0 & \gamma_4-1 & 0 \\ 0 & \gamma_2-1 & -1 & 0 & \gamma_5-1 \\ \gamma_1-1 & 0 & \gamma_3-1 & -1 & 0 \\ 0 & \gamma_2-1 & 0 & \gamma_4-1 & -1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \\ c_5 \end{bmatrix} = \begin{bmatrix} \Delta \\ \Delta \\ \Delta \\ \Delta \\ \Delta \end{bmatrix}$$

2. Characterize the regular time points and show that on these time points the Lyapounov function has negative drift
3. Now apply Jim Dai's stability framework

Non-Stabilizability

The Push-Pull Network with $\lambda_i = \mu_i$



Is there a $\mathcal{P} : \mathbb{Z}_+^2 \to \{\text{push}, \text{pull}\}^2$ (with restrictions at the axes) such that the Markov chain has a positive recurrent state that is reached w.p. 1?

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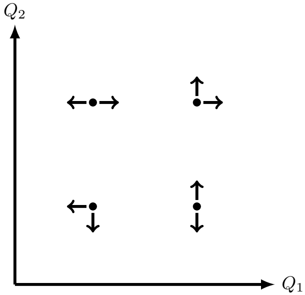

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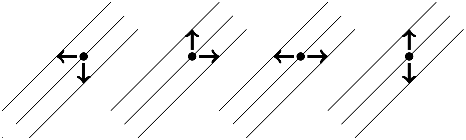
Theorem (Leonardo Rojas-Nandayapa, Tom Salisbury, Y.N.)

The push-pull network with $\lambda_i = \mu_i, i = 1, 2$ is non-stabilizable.

Non-stabilizability Proof

(This version assumes: $\lambda_1 = \mu_1 = \mu_2 = \lambda_2$) for simplicity

1. Set x_n as the embedded Markov chain of $X(t) = (Q_1(t), Q_2(t))$
2. Define $g\Big((x_1, x_2)\Big) = x_1 - x_2$ and $Z_n = g(X_n)$
   Z_n is a martingale for any $\mathcal{P}$:



3. Assume $\exists$ positive recurrent $\mathcal{B} \subset \mathbb{Z}_+^2$.
   Take $x, y \in \mathcal{B}$ with $g(x) \neq g(y)$
4. Set $X_0 = x$ and define $T = \inf\{n \geq 0 : X_n = y\}$
5. For $\mathcal{B}$ to be positive recurrent, $E[T] < \infty$ so:

$$g(x) = E[Z_0] = E[Z_T] = g(y),$$

a contradiction.

A More General Result for Non-Stabilizability

The idea of finding a linear-martingale simultaneously for all possible policies turns out to be fruitful in greater generality:

A More General Result for Non-Stabilizability

The idea of finding a linear-martingale simultaneously for all possible policies turns out to be fruitful in greater generality:

Theorem (Leonardo Rojas-Nandayapa, Tom Salisbury, Y.N.)

Consider controlled queueing networks $\{X_n, \ \ n \geq \ 0\}$ on $\mathbb{Z}_+^M$ with $L < \infty$ possible actions. Denote, by $\mathbf{D}$ the $L \times M$ matrix with rows,

$$\mathbf{\Delta}_i := E_{\text{action } i}[X_{n+1} - X_n \,|\, X_n], \quad i = 1, \ldots, L.$$

Then subject to technical non-degeneracy conditions, if

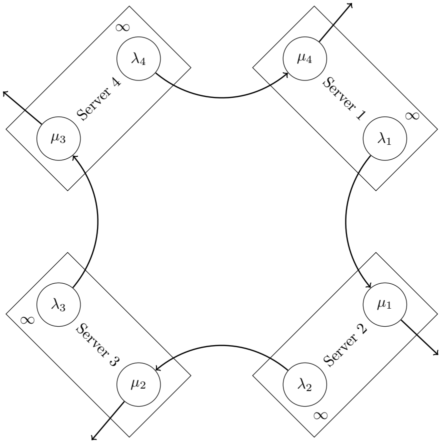
$$\text{rank}\big(\mathbf{D}\big) < M,$$

then the network is non-stabilizable.

Applications to Structured Networks

Corollary (Leonardo Rojas-Nandayapa, Tom Salisbury, Y.N.)

Push-Pull Rings with M **even** and $\lambda_i = \mu_i$ are non-stabilizable.



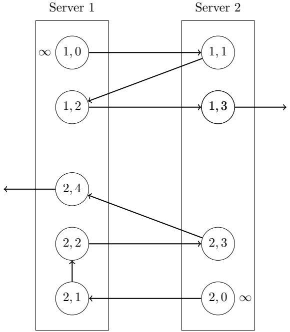
Applications to Structured Networks cont.

Corollary (Leonardo Rojas-Nandayapa, Tom Salisbury, Y.N.)

Consider an infinite supply network with 2 servers and S streams. Assume,

$$\sum_{j \in C_1(i)} \mu_{i,j}^{-1} = \sum_{j \in C_2(i)} \mu_{i,j}^{-1}, \quad i = 1, \ldots, S,$$

then the network is non-stabilizable.



Wrap-up

Current Projects Related to Stability Properties of Queueing Networks

- With Leonardo Rojas-Nandayapa and Tom Salisbury: Non-stabilizability of similar models under general processing time assumptions (can not use Martingale method as is)
- With Erjen Lefeber and Dieter Armbruster: It is known that in certain cases stability depends on the distributional assumptions. We have an illustration of this phenomenon based on deterministic dynamical (hybrid) systems
- With Gideon Weiss and Erjen Lefeber: General stability results for general queueing networks with infinite supplies
- Long term interest: Designing and understanding stabilizing adaptive control methods for complex queueing networks