# Finite Buffer Queueing/Fluid Networks with Overflows

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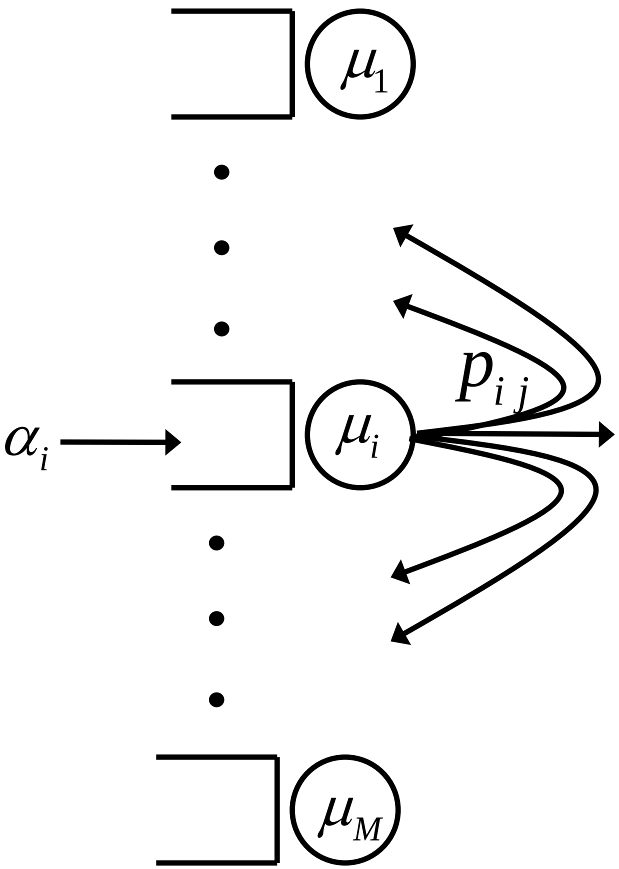
# Background Jackson Networks and LCP

# Open Jackson Networks

Jackson 1957, Goodman & Massey 1984, Chen & Mandelbaum 1991




$$\bar{p}_i = 1 - \sum_{j=1}^{M} p_{ij}$$

**Problem Data:**

$$\mu_M, \alpha_M, P_{M \times M}$$

Assume: open, no "dead" nodes

**Traffic Equations (Stable Case):**

$$\lambda_i = \alpha_i + \sum_{j=1}^{M} \lambda_j p_{ji}$$

$$\lambda = \alpha + P'\lambda$$

$$\lambda = (I - P')^{-1}\alpha$$

**Traffic Equations (General Case):**

$$\lambda_i = \alpha_i + \sum_{j=1}^{M} \left(\lambda_j \wedge \mu_j\right) p_{ji}$$

$$\lambda = \alpha + P'\left(\lambda \wedge \mu\right)$$

$$LCP\left(\ \alpha - (I - P')\mu\ ,\ (I - P')\ \right)$$

# The Linear Complementarity Problem (LCP)

$q \in \mathbb{R}^n, M \in \mathbb{R}^{n \times n}$

$$LCP(q, M): Find\ z, w \in \mathbb{R}^n:$$
$$w - Mz = q,$$
$$w \geq 0, z \geq 0,$$
$$w'z = 0.$$

The last (complemenatrity) condition reads:

$w_i > 0 \Rightarrow z_i = 0$ and $z_i > 0 \Rightarrow w_i = 0.$

# Min-Linear Equations (Using LCP)

Find $\lambda$ : $\boxed{\lambda = \gamma + B(\lambda \wedge \mu)}$

$\delta = \lambda \wedge \mu$

$$\lambda = \gamma + B\delta$$
$$0 \le \delta \le \lambda$$
$$0 \le \delta \le \mu$$
$$(\lambda - \delta)'(\mu - \delta) = 0$$

$w = \lambda - \delta, z = \mu - \delta$

$$w - (I - B)z = \gamma - (I - B)\mu$$
$$z \ge 0, w \ge 0$$
$$w'z = 0$$

$$\boxed{LCP(\gamma - (I - B)\mu, I - B)}$$

# Classic Product Form Results

Jackson 1957, Goodman & Massey 1984

**Again the Traffic Equations :**

$$\lambda_i = \alpha_i + \sum_{j=1}^{M} \left(\lambda_j \wedge \mu_j\right) p_{ji}$$

$$\lambda = \alpha + P'\left(\lambda \wedge \mu\right)$$

$$LCP\Big(\ \alpha - (I - P')\mu\ ,\ (I - P')\ \Big)$$

Assume arrivals are Poisson processes and i.i.d. exponential service durations

Jackson 57': When $\lambda_i < \mu_i$ $i = 1, \ldots, M$:

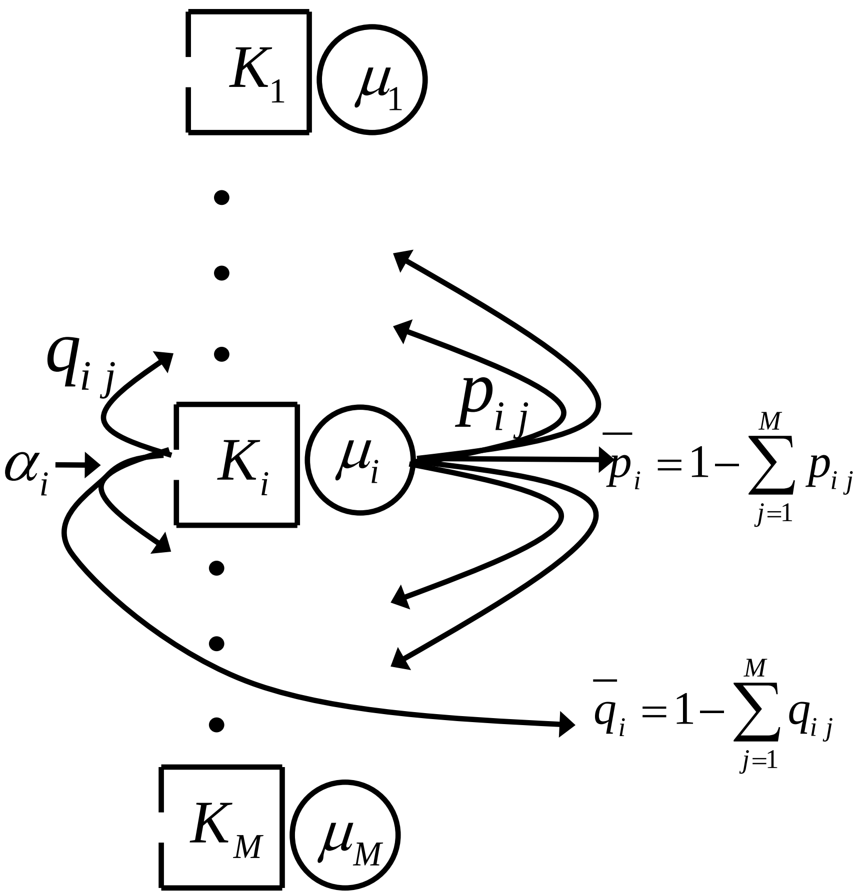
$$\lim_{t\to\infty} P(X_1(t) = x_1, \ldots, X_M(t) = x_M) = \prod_{i=1}^{M} \left(1 - \frac{\lambda_i}{\mu_i}\right)\left(\frac{\lambda_i}{\mu_i}\right)^{x_i}$$

G&M 84': On the set of non-overloaded nodes product form holds ($\lambda_i < \mu_i$) and the other nodes diverge to infinity a.s.

# Our Contribution:
# Finite Buffers with Overflows

# Modification: Finite Buffers and Overflows

Practically important but not as tractable




**Problem Data:**

$$\mu_M\,,\ \alpha_M\,, P_{M\times M}\,, K_M\,, Q_{M\times M}$$

Assume: open, no “dead” nodes, no “jam” (open overflows)

**Exact Traffic Equations:**

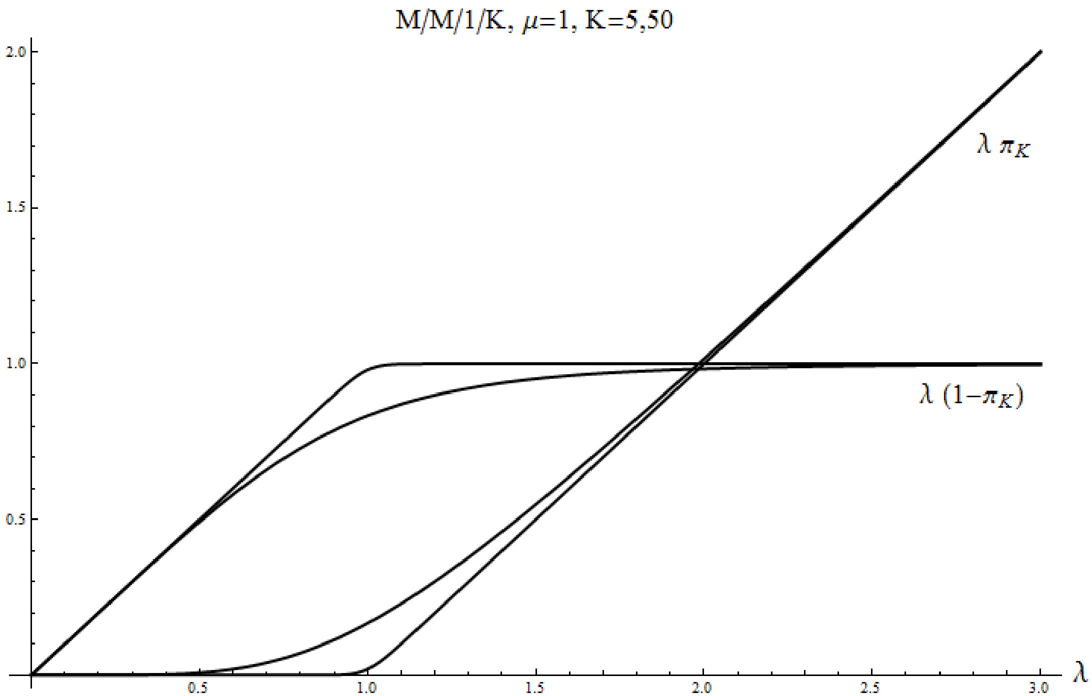
**Generally No**

**Explicit Solutions:**

**Generally No**

Nico van Dijk, 1988. Yes if P=Q.

# When K is Big, Things are "Simpler"




For K big:

$$\text{out rate} \approx \lambda \wedge \mu$$

$$\text{overflow rate} \approx \lambda - \lambda \wedge \mu = (\lambda - \mu)^{+}$$

So scale the system with $N = 1, 2, \ldots$ :

$$\alpha^{N} = \mathrm{N}\,\alpha$$

$$\mu^{N} = N\mu$$

$$\mathrm{K}^{N} = N\,K$$

# Limiting Traffic Equations

$$\text{limiting out rate} = \lambda \wedge \mu$$

$$\text{limiting overflow rate} = (\lambda - \mu)^{+}$$

$$\lambda_i = \alpha_i + \sum_{j=1}^{M} \left( \lambda_j \wedge \mu_j \right) p_{ji} + \sum_{j=1}^{M} \left( \lambda_j - \mu_j \right)^{+} q_{ji}$$

or

$$\lambda = \alpha + P'\left( \lambda \wedge \mu \right) + Q'(\lambda - \mu)^{+}$$

$$\lambda = \alpha + \mathrm{P}'(\lambda \wedge \mu) + Q'(\lambda - \lambda \wedge \mu)$$

$$\lambda = (I - Q')^{-1}\alpha + (I - Q')^{-1}(P' - Q')(\lambda \wedge \mu)$$

or

$$LCP\left( (I - Q')^{-1}(\alpha - (I - P')\mu)\ ,\ (I - Q')^{-1}(I - P') \right)$$

# Digression:
# The Linear Complementarity Problem (LCP)

# The Linear Complementarity Problem (LCP)

$q \in \mathbb{R}^n, M \in \mathbb{R}^{n\times n}$

$$\begin{aligned} LCP(q,M): Find\ z,w \in \mathbb{R}^n:& \\ & w - Mz = q, \\ & w \geq 0, z \geq 0, \\ & w'z = 0. \end{aligned}$$

The last (complemenatrity) condition reads:

$w_i > 0 \Rightarrow z_i = 0$ and $z_i > 0 \Rightarrow w_i = 0.$

# It's all about Choosing a Subset…

For $\alpha \subseteq \{1,...,n\}$ denote by $B(\alpha)$ a matrix with collumns $\alpha$ taken from $I$ (identity matrix) and collumns $\{1,...,n\} \setminus \alpha$ taken from $-M$.

$LCP$ is about finding $\alpha$ and $x \geq 0$ such that

$$B(\alpha)\,x = q$$

In this case:

$$w_i = \begin{cases} x_i & i \in \alpha \\ 0 & i \notin \alpha \end{cases}, \qquad z_i = \begin{cases} 0 & i \in \alpha \\ x_i & i \notin \alpha \end{cases}.$$

# Illustration: n=2

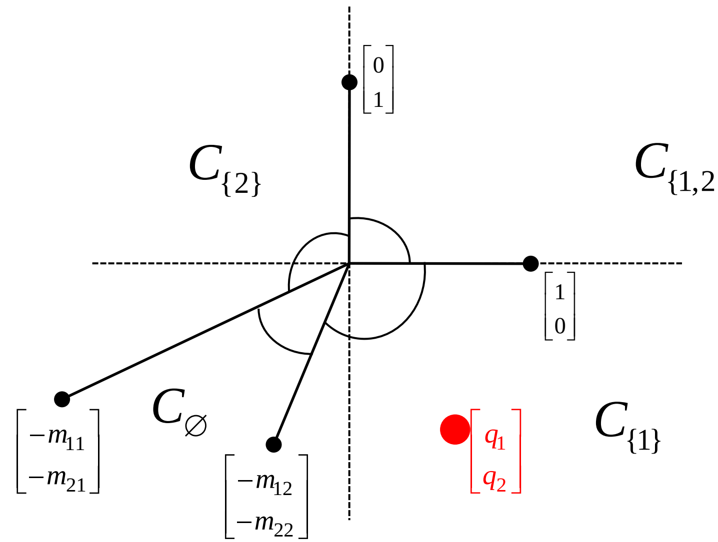
$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} w_1 \\ w_2 \end{bmatrix} - \begin{bmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{bmatrix}\begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \begin{bmatrix} q_1 \\ q_2 \end{bmatrix}$$

$$\alpha = \{1,2\}: \quad \begin{bmatrix} 1 \\ 0 \end{bmatrix} w_1 + \begin{bmatrix} 0 \\ 1 \end{bmatrix} w_2 = \begin{bmatrix} q_1 \\ q_2 \end{bmatrix}$$

$$\alpha = \{1\}: \quad \begin{bmatrix} 1 \\ 0 \end{bmatrix} w_1 + \begin{bmatrix} -m_{12} \\ -m_{22} \end{bmatrix} z_2 = \begin{bmatrix} q_1 \\ q_2 \end{bmatrix}$$

$$\alpha = \{2\}: \quad \begin{bmatrix} -m_{11} \\ -m_{21} \end{bmatrix} z_1 + \begin{bmatrix} 0 \\ 1 \end{bmatrix} w_2 = \begin{bmatrix} q_1 \\ q_2 \end{bmatrix}$$

$$\alpha = \varnothing: \quad \begin{bmatrix} -m_{11} \\ -m_{21} \end{bmatrix} z_1 + \begin{bmatrix} -m_{12} \\ -m_{22} \end{bmatrix} z_2 = \begin{bmatrix} q_1 \\ q_2 \end{bmatrix}$$




**Complementary cones:**

$$C_\alpha = \{y : y = B(\alpha)u, u \geq 0\}$$

**Immediate naïve algorithm with complexity** $2^n n^3$ *or* $2^n + n^3$

# Existence and Uniqueness

Definition: A matrix, $M \in \mathbb{R}^{n\times n}$ is a P-matrix if the determinants of all $(2^n - 1)$ principal submatrices are positive.

Theorem (1958): $LCP(q, M)$ has a unique solution for all $q \in \mathbb{R}^n$ if and only if $M$ is a P-matrix.

e.g. for $n = 2$: $m_{11} > 0,\ m_{22} > 0,\ m_{11}m_{22} - m_{12}m_{21} > 0$

P-matrix means that the complementary cones "parition" $\mathbb{R}^n$

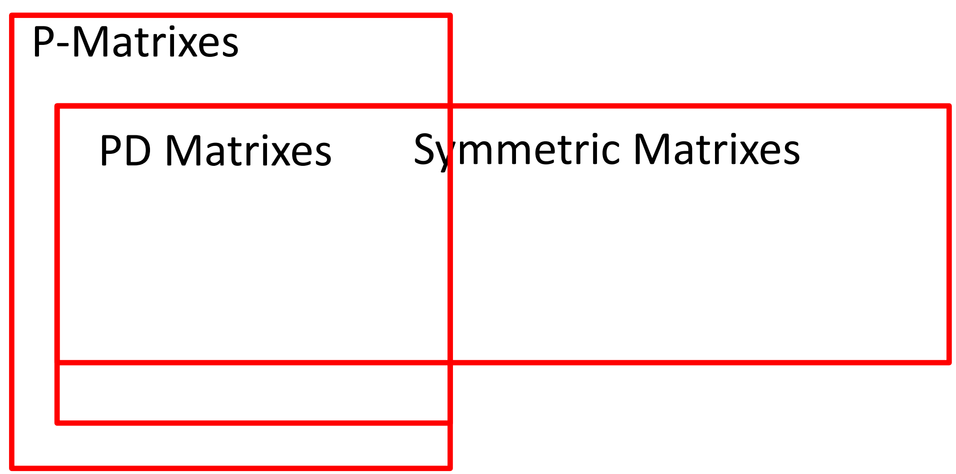
**Relation of P-matrixes to positive definite (PD) matrixes:**

Reminder(PD):

$x' M x > 0 \quad \forall x \neq 0$

Reminder(PSD):

$x' M x \geq 0 \quad \forall x$



# Computation (Algorithms)

- Naive algorithm, runs on all subsets alpha (intractable)
- Generally, LCP is NP complete
- Lemeke's Algorithm, a bit like simplex
- If M is PSD: polynomial time algorithms exists
- PD LCP equivalent to QP
- Special cases of M, linear number of iterations
- Note: Checking for P-Matrix is NP complete, checking for PD is polynomial time
- For our special case we have an algorithm with a quadratic number of iterations

  (Still have not done: proven uniqueness using LCP theory).

**Show:** $(I - Q')^{-1}(I - P')$ is a P matrix

# How does LCP generalize LP and QP?

# Linear Programming (LP)

Primal-LP:

$$\begin{aligned} \min \quad & c'x \\ s.t. \quad & Ax \geq b \\ & x \geq 0 \end{aligned}$$

Dual-LP:

$$\begin{aligned} \max \quad & b'y \\ s.t. \quad & A'y \leq c \\ & y \geq 0 \end{aligned}$$

$$\begin{aligned} \min \quad & c'x \\ s.t. \quad & Ax - b = v \\ & x, v \geq 0 \end{aligned}$$

$$\begin{aligned} \max \quad & b'y \\ s.t. \quad & u = c - A'y \\ & y, u \geq 0 \end{aligned}$$

**Theorem: Complementary slackness conditions**

Assume $x, v, y, u$ are feasible for primal and dual:

$x_i u_i = 0, \quad y_i v_i = 0 \quad \Leftrightarrow \quad$ They are optimal solutions

# The LCP of LP

Find: $u, v, x, y \geq 0$

Such that:

$$\begin{bmatrix} u \\ v \end{bmatrix} - \begin{bmatrix} 0 & -A' \\ A & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} c \\ -b \end{bmatrix}$$

And (complementary slackness):

$$u'x = 0 \quad v'y = 0$$

$$LCP\left( \begin{bmatrix} c \\ -b \end{bmatrix}, \begin{bmatrix} 0 & -A' \\ A & 0 \end{bmatrix} \right)$$

# Quadratic Programming

QP:
$$\min \quad Q(x) = c'x + \frac{1}{2}x'Dx$$
$$s.t. \quad Ax \geq b$$
$$x \geq 0$$

**Lemma: An optimizer, $\bar{x}$, of the QP also optimizes**

QP-LP:
$$\min \quad (c + D\bar{x})'x$$
$$s.t. \quad Ax \geq b$$
$$x \geq 0$$

Proof:

Let $\hat{x}$ be feasible. $0 < \eta < 1$, $x_\eta = \bar{x} + \eta(\hat{x} - \bar{x})$

$$Q(x_\eta) - Q(\bar{x}) \geq 0$$

$$(c' + D\bar{x})'(\hat{x} - \bar{x}) \geq \frac{-\eta}{2}(\hat{x} - \bar{x})'D(\hat{x} - \bar{x})$$

$$(c' + D\bar{x})'(\hat{x} - \bar{x}) \geq 0$$

$$(c' + D\bar{x})'\hat{x} \geq (c' + D\bar{x})'\bar{x}$$

■

**QP-LP gives a necessary condition for optimality of QP in terms of an checking optimality of an LP**

# The Resulting LCP of QP

$$LCP\left(\begin{bmatrix} c \\ -b \end{bmatrix}, \begin{bmatrix} D & -A' \\ A & 0 \end{bmatrix}\right)$$

**Allows to find "suspect" points that satisfy the necessary conditions: QP-LP**

**Theorem: Solutions of this LCP are KKT (Karush-Kuhn-Tucker) points for the QP**

Proof: Write down KKT conditions and check.

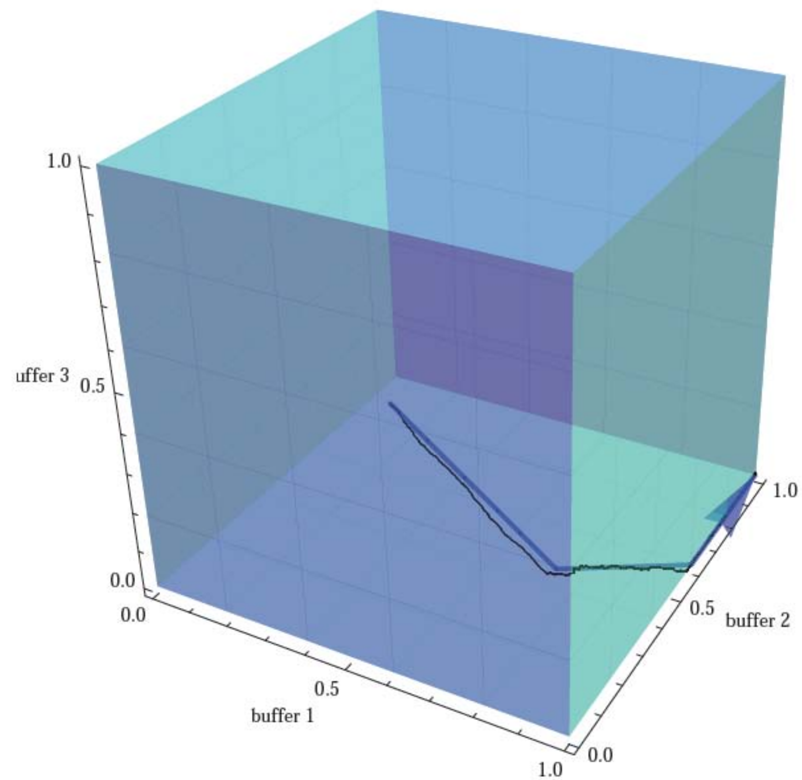
**Corollary: If D is PSD then x solving the LCP optimizes QP.**

**Note: When D is PSD then M is PSD. In this case it can be shown that the LCP is equivalent to a QP (solved in polynomial time). Similarly, every PSD LCP can be formulated as a PSD QP.**

# Back To Our Problem:
# The Fluid Network

# Limiting Trajectories

In similar spirit to the traffic equations, limiting trajectories, $x(t)$, may be calculated…




**We think:**

$$\lim_{N\to\infty} \sup_t \left\{ \left| \frac{X^N(t)}{N} - x(t) \right| \right\} = 0 \quad \text{a.s.}$$

# Sojourn Times

$$\text{Sojourn Time} \equiv \text{Time in system of customer arriving to steady state FCFS system}$$

$$S^N \equiv \text{Sojourn time of customer in } N\text{'th scaled system}$$

We want to find the limiting distribution of $S^N$

# Construction of Limiting Sojourn Times

$$F = \{1, \ldots, s\} \qquad \lambda_i > \mu_i \ \ for\, i \in S$$

$$\overline{F} = \{s+1, \ldots, M\} \qquad \lambda_i < \mu_i \ \ for\, i \in \overline{S}$$

Observe,

$$\text{time through i} \in \text{F} \ \approx \frac{K_i}{\mu_i} \qquad \text{time through i} \in \overline{\text{F}} \ \approx 0$$

For job at entrance of buffer $\text{i} \in \text{F}$

$$w.p. \approx \frac{\mu_i}{\lambda_i} \text{ enters buffer i}$$

$$w.p. \approx \left(1 - \frac{\mu_i}{\lambda_i}\right) q_{ij} \text{ routed to entrace of buffer j}$$

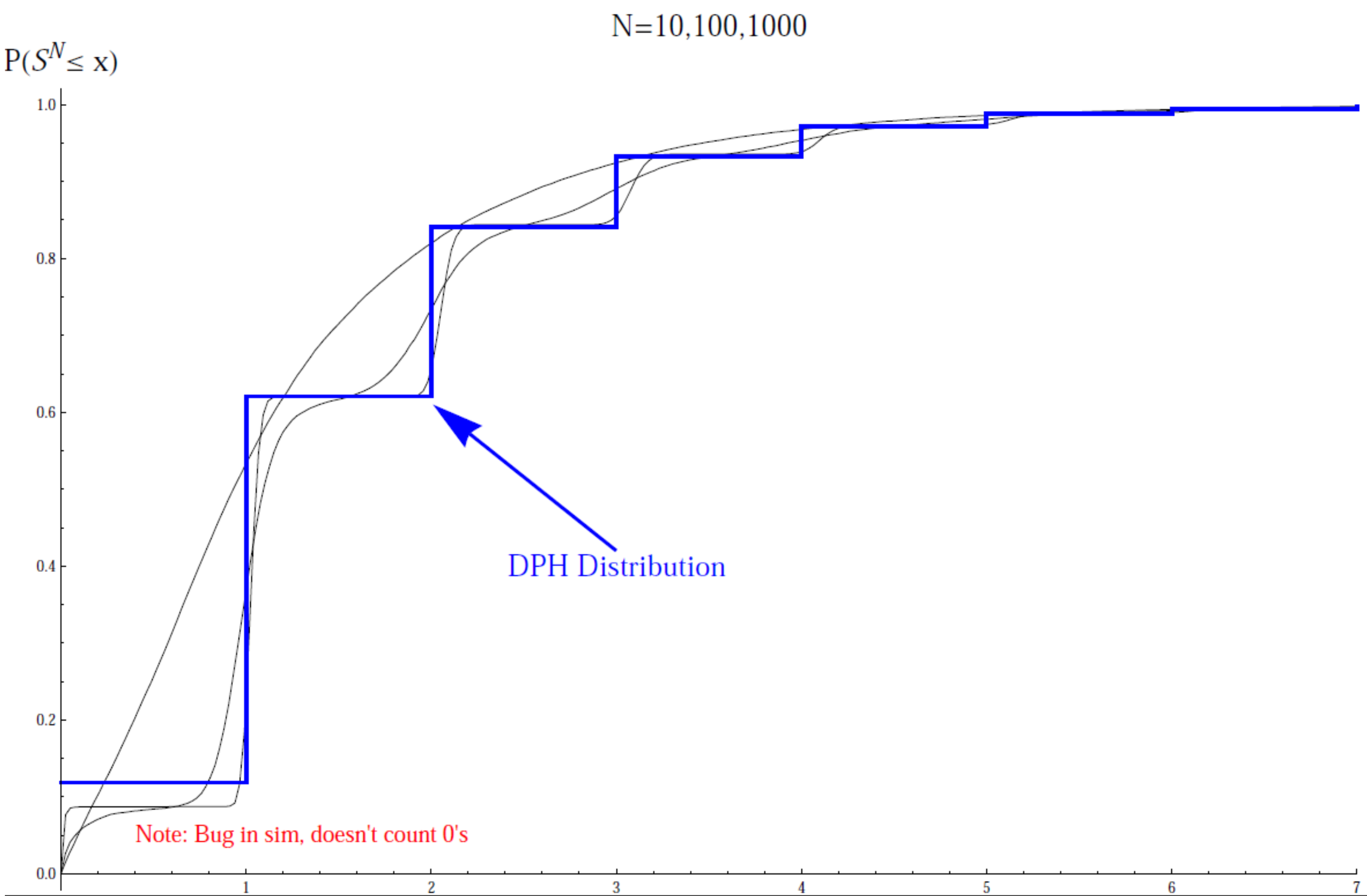
$$w.p. \approx \left(1 - \frac{\mu_i}{\lambda_i}\right) \overline{q}_i \text{ leaves the system}$$

A job at entrance of buffer $\text{i} \in \overline{\text{F}}$ routed almost immediately according to $P$

A "fast" chain and "slow" chain…

# Sojourn Times Scale to a Discrete Distribution!!!

**We think:** $K_i / \mu_i = 1, \; i \in F \qquad S^N \Rightarrow DPH\left(T_{s \times s}, \tau_{1 \times s}\right)$



# The "Fast" Chain and "Slow" Chain

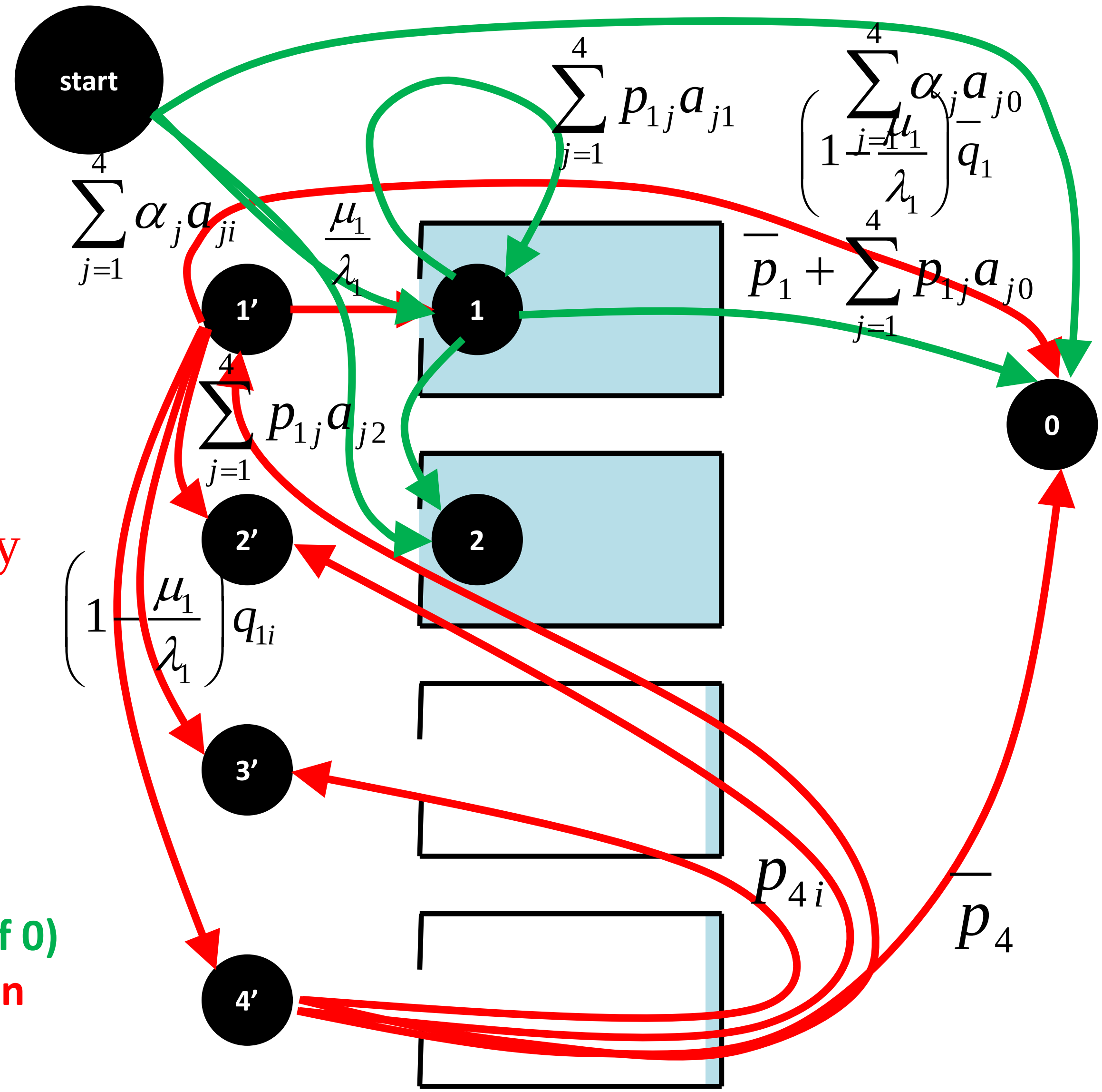
Example: $M = 4$,

$\frac{K_1}{\mu_1} = \frac{K_2}{\mu_2} = 1, \sum_{i=1}^{4} \alpha_i = 1,$

$F = \{1, 2\}, \overline{F} = \{3, 4\}$:

**"Fast" chain on {0, 1, 2, 1', 2', 3', 4'}:**

$a_{ij} \equiv$ Absorbtion probability in $j \in \{0,1,2\}$ starting in i'




**"Slow" chain on {0, 1, 2}**

**DPH distribution (hitting time of 0) transitions based on "Fast" chain**

# The DPH Parameters (Details)

$$F = \{1, \ldots, s\},\ \overline{F} = \{s+1, \ldots, M\}$$

## "Fast" chain

$$B_{M\times s} = \begin{bmatrix} \mu_1/\lambda_1 & & 0 \\ & \ddots & \\ 0 & & \mu_s/\lambda_s \\ & 0_{M-s\times s} & \end{bmatrix} \qquad C_{M\times M} = \begin{bmatrix} 1-\frac{\mu_1}{\lambda_1} & & & \\ & \ddots & & 0_{M\times M-s} \\ & & 1-\frac{\mu_s}{\lambda_s} & \\ & 0_{M-s\times s} & & \end{bmatrix} \cdot Q + \begin{bmatrix} 0_{M\times s} & \begin{matrix} 0_{s\times M-s} \\ I_{M-s} \end{matrix} \end{bmatrix} \cdot P$$

$$A_{M\times s} = (I - C)^{-1} \cdot B$$

## "Slow" chain

$$T_{s\times s} = \begin{bmatrix} I_s & 0_{s\times M-s} \end{bmatrix} \cdot P \cdot A \qquad \tau_{s\times 1} = \frac{1}{\sum_{j=1}^{M} \alpha_j} \alpha^T \cdot A$$

$$S \sim DPH(T_{s\times s}, \tau_{1\times s}) \qquad \mathrm{P}(S \le k) = 1 - \tau \cdot T^k \cdot 1_{s\times 1}$$

# Mechanism of nonlinear flow pattern selection in moderately non-Boussinesq mixed convection

Yoni Nazarathy, Sergey Suslov,
John Beynon, William Phillips

Swinburne Applied Mathematics Seminar,
April **1**, 2011.