Scheduling Arrivals to a Processor Sharing Congestion System with Linear Slowdown

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n users interact

$$1 = \int_{a_i}^{d_i} v\big(q(t)\big)\,dt \qquad q(t) = \sum_{j=1}^{n} \mathbb{1}\{t \in [a_j, d_j]\}$$

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Objective

$$\min_{\mathbf{a}\in\mathbb{R}^n} c(\mathbf{a}) = \sum_{i=1}^{n} c_i\big(a_i, d_i(\mathbf{a})\big), \qquad c_i(a_i, d_i) = (d_i - d_i^*)^2 + \gamma\,(d_i - a_i)$$

Objective (simple version)

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Objective (complex version)

$$\begin{aligned} c_i(a_i, d_i) = {} & g_i^{(1)}\Big((d_i - d_i^*)^+\Big) + g_i^{(2)}\Big((d_i^* - d_i)^+\Big) \\ & + g_i^{(3)}\Big((a_i - a_i^*)^+\Big) + g_i^{(4)}\Big((a_i^* - a_i)^+\Big) \\ & + g_i^{(5)}\Big(d_i - a_i\Big) \end{aligned}$$

Example: $n = 15$, $\alpha = \frac{0.8}{n}$, $\beta = 1$ and $\mathbf{d}^*$ quantiles of Normal$(0, \frac{1}{4})$

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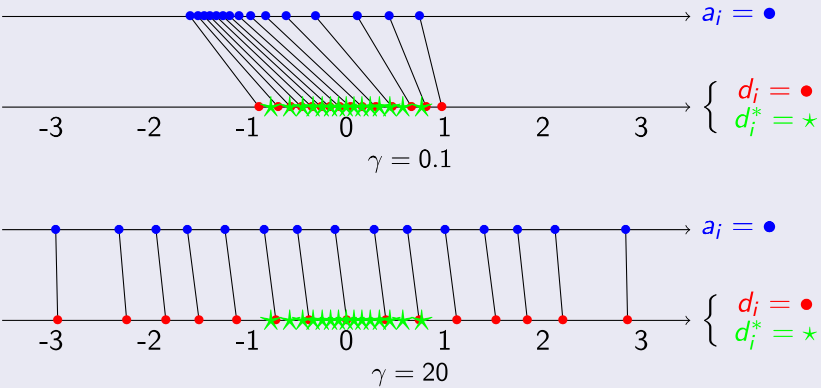
Optimal Schedules



Basic Equations

Observation

If we order d_i^* then a_i and d_i follow the same order

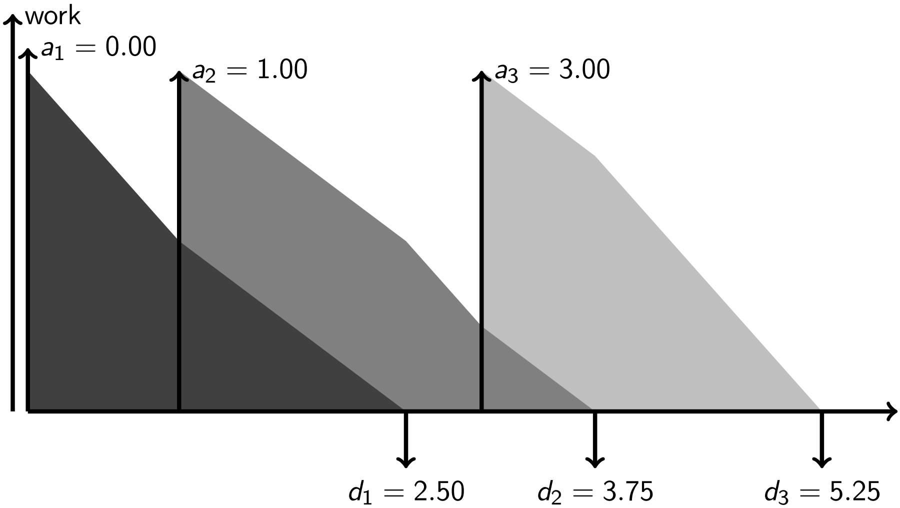
$$
\begin{aligned}
1 &= \int_{a_i}^{d_i} \beta - \alpha\Big(\sum_{j=1}^{n} \mathbb{1}\{t \in [a_j, d_j]\} - 1\Big) dt \\
&= (\beta + \alpha)(d_i - a_i) - \alpha \sum_{j=1}^{n} (d_i \wedge d_j - a_i \vee a_j)^+ \\
&= \dots \\
&= -(\beta - \alpha(i - h_i))a_i + (\beta - \alpha(k_i - i))d_i - \alpha\Big(\sum_{j=h_i}^{i-1} d_j - \sum_{j=i+1}^{k_i} a_j\Big)
\end{aligned}
$$

with "encodings" of the interleaving of $\mathbf{d}$ and $\mathbf{a}$:

$$
k_i := \max\big\{k \;:\; a_k \le d_i\big\}, \qquad h_i := \min\big\{h \;:\; d_h \ge a_i\big\}
$$

Piecewise affine relationship between **a** and **d**

E.g. with $n = 3$, $\beta = 1/2$, $\alpha = 1/6$:



Key attributes and algorithms

An exponential number of convex quadratic programs

The objective function is piecewise quadratic with number of regions equal to,

$$\frac{\binom{2n}{n}}{n+1} \sim \frac{4^n}{n^{3/2}\sqrt{\pi}},$$

and with explicit expressions for describing each of the QPs.

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4. Efficient trajectory calculation for $\mathbf{d}(a_i)$

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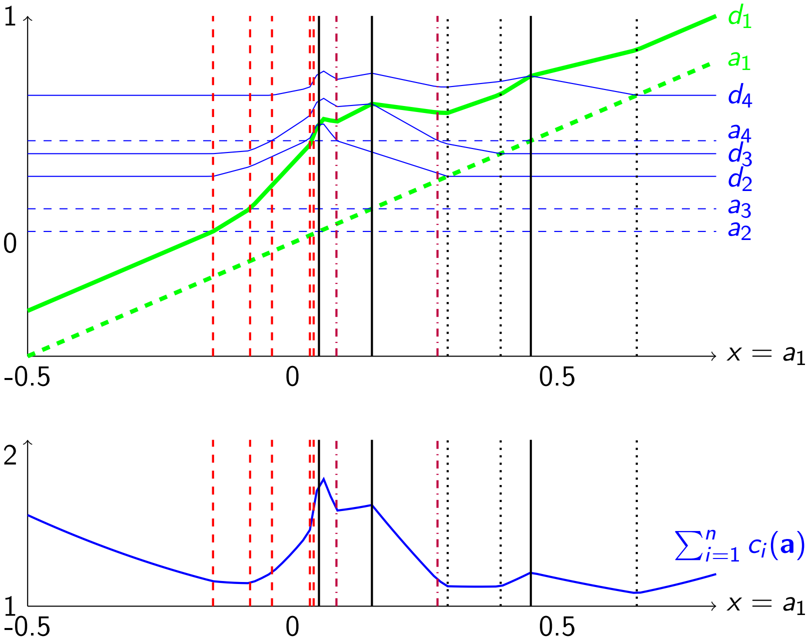
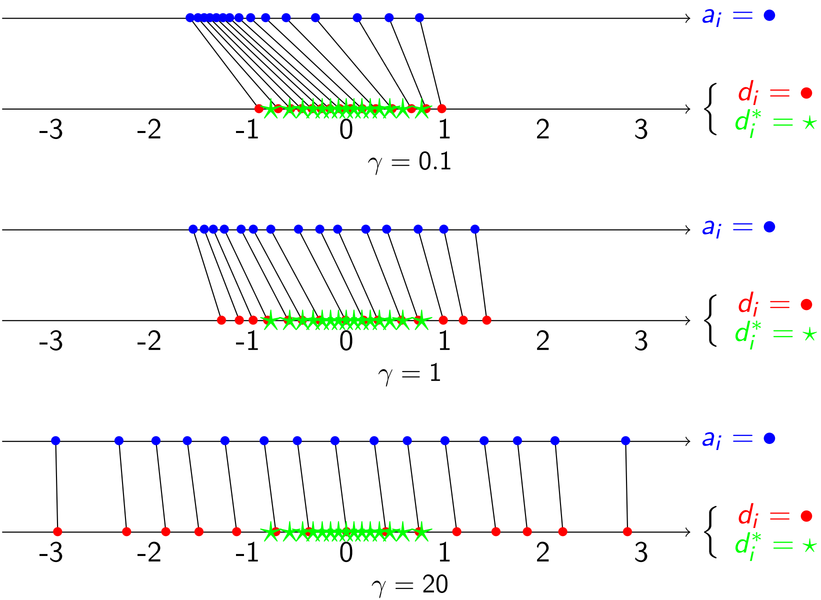
Not known if problem is NP–complete

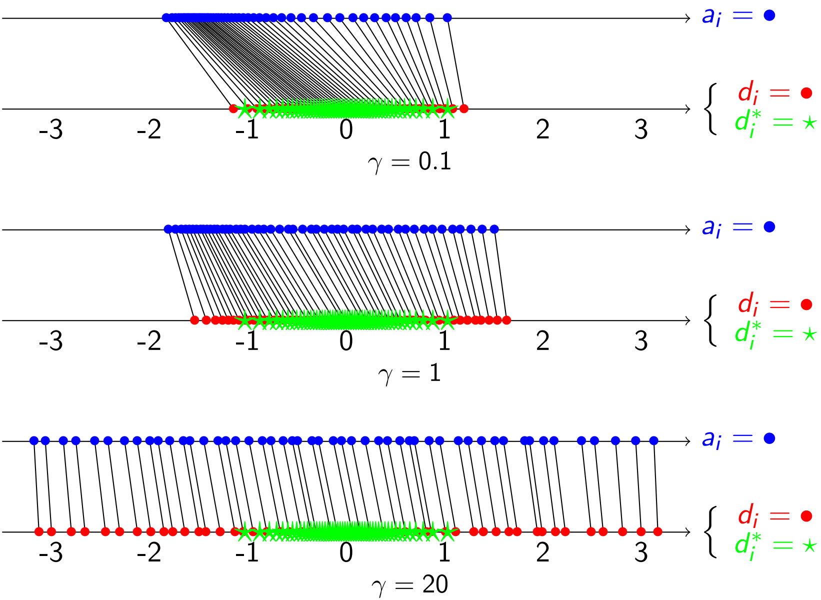
Trajectory of $\mathbf{d}$ when changing a_1 ($\alpha = 1.5$, $\beta = 5$ and $\mathbf{d}^* = \mathbf{0}$)



Optimal dynamics $n = 15$ ($\alpha = \frac{0.8}{n}$, $\beta = 1$ and $\mathbf{d}^*$ quantiles of Normal$(0, \frac{1}{4})$)



Heuristic dynamics $n = 50$ ($\alpha = \frac{0.8}{n}$, $\beta = 1$ and $\mathbf{d}^*$ quantiles of Normal$(0, \frac{1}{4})$)



More General Dynamics... (not yet handled but appears possible)

A tandem sequence of K processors

$$\ell_i^k = \int_{a_i^{k-1}}^{a_i^k} v_i^k\big(q^k(t)\big)\, dt, \quad k = 1, \ldots, K$$

$$q^k(t) = \sum_{j=1}^{n} \mathbb{1}\{t \in [a_j^{k-1}, a_j^k]\}$$

$$v_i^k(q) = \beta_i^k - \alpha_i^k(q-1) = \text{or even arbitrary}$$

Look at start and end of the sequence: $a_i = a_i^0$, $d_i = a_i^K$

$$\min_{\mathbf{a}\in\mathbb{R}^n} c(\mathbf{a}) = \sum_{i=1}^{n} c_i\big(a_i, d_i(\mathbf{a})\big), \qquad c_i(a_i, d_i) = (d_i - d_i^*)^2 + \gamma\,(d_i - a_i)$$

Belief: Similar problem structure - maybe proving NP completeness is easier in this general case...

L. Ravner and Y. Nazarathy, "*Scheduling for a Processor Sharing System with Linear Slowdown*", preprint, arXiv:1508.03136, 2015.

L. Ravner, M. Haviv, H. Vu, "*A strategic timing of arrivals to a linear slowdown processor sharing system*", preprint, arXiv:1508.03420, 2015.