

Non-existence of stable policies for critical queueing networks with infinite supplies

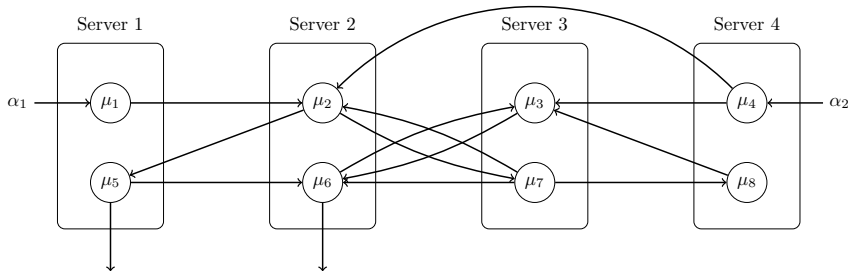
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Joint work with Leonardo Rojas-Nandayapa and Tom Salisbury

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Multi-Class Queueing Networks: (MCQN, 1990's ...)



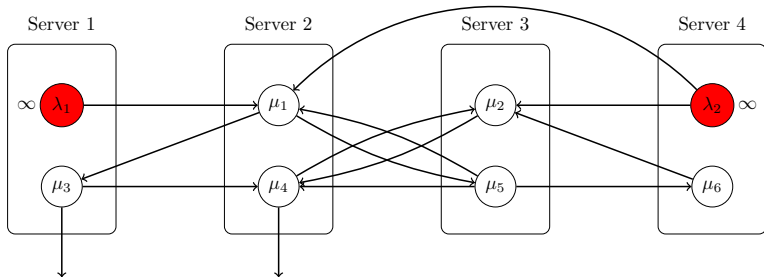
Major Question: How to control for stability?

Bramson, Dai, Hasenbein, Kumar, Meyn, Stolyar, Weiss ...

Full utilization \Rightarrow Queue blow-up

Diffusion Limits: Bramson, Dai, Harrison, Reiman, Williams ...

Queueing networks with infinite supplies: (MCQN-IVQ, Weiss et. al.)



Useful models: Full utilization without queue blow-up

classes: $\mathcal{K} = \mathcal{K}_a \cup \mathcal{K}_s$

servers: \mathcal{I} , with $\mathcal{C}(i) = \{k \in \mathcal{K} : k \text{ served by } i\}$

SLLN for primitive counting processes:

$$S_k(t) \sim \lambda_k t, \quad k \in \mathcal{K}_a$$

$$S_k(t) \sim \mu_k t, \quad k \in \mathcal{K}_s$$

$$\Phi_{k,k'}(\ell) \sim p_{k,k'} \ell, \quad k \in \mathcal{K}, k' \in \mathcal{K}_s$$

Dynamics $(Q(\cdot), T(\cdot))$:

$$Q_k(t) = Q_k(0) + \sum_{k' \in \mathcal{K}} \Phi_{k',k}(S_{k'}(T_{k'}(t))) - S_k(T_k(t)), \quad k \in \mathcal{K}_s$$

$$Q_k(t) = Q_k(0) + \sum_{k' \in \mathcal{K}} \Phi_{k',k} \left(S_{k'}(T_{k'}(t)) \right) - S_k(T_k(t)), \quad k \in \mathcal{K}_s$$

constraints:

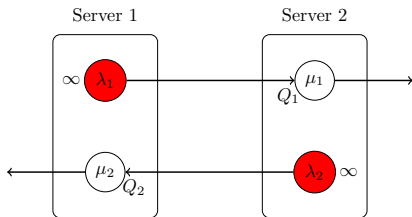
$$\int_0^t \mathbb{1}\{Q_k(s) = 0\} dT_k(s) = 0, \quad k \in \mathcal{K}_s$$

$$\sum_{k \in \mathcal{C}(i)} T_k(t) \leq t, \quad \forall i \in \mathcal{I}$$

For some servers, want full utilization:

$$\sum_{k \in \mathcal{C}(i)} T_k(t) = t$$

e.g. “push-pull” network:

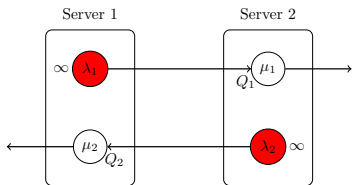


$$Q_1(t) = Q_1(0) + S_{a,1}(T_1(t)) - S_{s,1}(t - T_2(t))$$

$$Q_2(t) = Q_2(0) + S_{a,2}(T_2(t)) - S_{s,2}(t - T_1(t))$$

$$T_1(t) \geq \int_0^t \mathbb{1}\{Q_2(s) = 0\} ds, \quad T_2(t) \geq \int_0^t \mathbb{1}\{Q_1(s) = 0\} ds,$$

Want an adapted policy (yielding $T(\cdot)$), such that $Q(\cdot)$ is stable

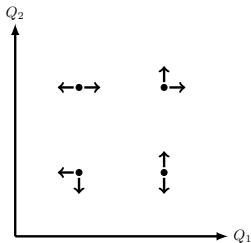


Control policy

$$\mathcal{P} : \mathbb{Z}_+^2 \rightarrow \{ \text{"do } \lambda", \text{"do } \mu" \}^2$$

with (axis restrictions):

$$\mathcal{P}(\cdot, 0) = (\text{"do } \lambda", \cdot), \quad \mathcal{P}(0, \cdot) = (\cdot, \text{"do } \lambda")$$



Stability :

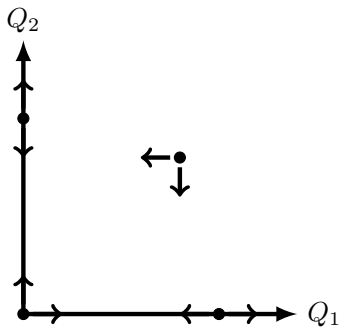
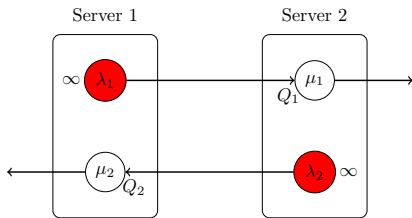
$\exists \mathcal{A} \subset \mathbb{Z}_+^{|\mathcal{K}_s|}$, with $|\mathcal{A}| < \infty$ such that for all $Q(0) \in \mathbb{Z}_+^{|\mathcal{K}_s|}$,

$$\mathbb{E}[\inf\{t : Q(t) \in \mathcal{A}\}] < \infty$$

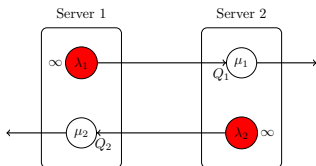
Assume $\lambda_i < \mu_i$ and take priority policy:

$$T_1(t) = \int_0^t \mathbb{1}\{Q_2(s) = 0\} ds, \quad T_2(t) = \int_0^t \mathbb{1}\{Q_1(s) = 0\} ds,$$

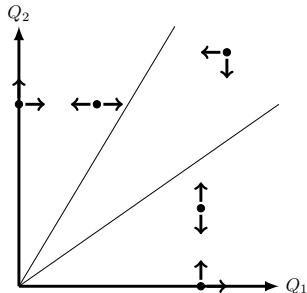
In this case the network is stable



Consider now the case, $\lambda_i > \mu_i$

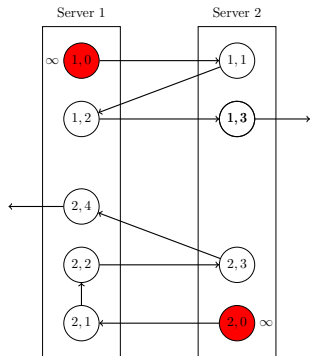
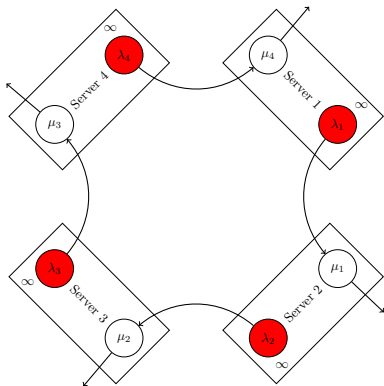


A threshold policy is stabilizing:



Some success with other structured networks also...

(Weiss et. al.)



also Re-entrant lines, etc...

Static Planning:

$$\begin{aligned} \max_{u,v} \quad & \sum_{i \in \mathcal{K}_a} g_i \lambda_i v_i \\ \text{s.t.} \quad & (I - P'_{s,s}) \text{diag}(\mu) u - P'_{a,s} \text{diag}(\lambda) v = \mathbf{0}, \\ & C_s u + C_a v \leq \mathbf{1}, \\ & u, v \geq \mathbf{0}. \end{aligned}$$

Solution (u^*, v^*) yields load vectors:

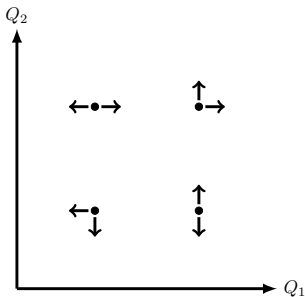
$$\rho := C_s u^* + C_a v^*, \quad \tilde{\rho} := C_s u^*$$

Question: Assume $\rho = \mathbf{1}$ but $\tilde{\rho} < \mathbf{1}$. Is there a policy that achieves u^*, v^* and is stable?

We believe the answer is almost always “yes”,
(ongoing work of Lefeber, Weiss, et. al to find a “universal policy”)

**In the current work we show non-existence of
stable policies for singular cases**

Consider now again the push-pull example with $\lambda_i = \mu_i$
(for simplicity assume $= 1$)

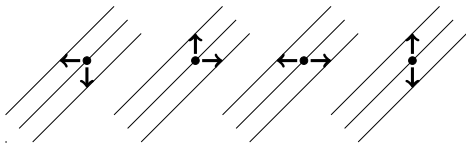


Is there a \mathcal{P} such that the network is stable?

Answer: No

Proof (push-pull exponential processing times)

- 1 $X(n) = (X_1(n), X_2(n))$ the embedded Markov chain of $(Q_1(t), Q_2(t))$
- 2 $g(x_1, x_2) = \lambda_2 x_1 - \lambda_1 x_2$ and $Z(n) = g(X_1(n), X_2(n))$
 $Z(\cdot)$ is a martingale for any policy \mathcal{P} :

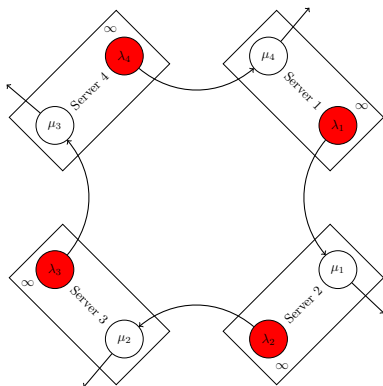


- 3 Assume \exists positive recurrent $\mathcal{A} \subset \mathbb{Z}_+^2$.
Take $x, y \in \mathcal{A}$ with $g(x) \neq g(y)$
- 4 Set $X(0) = x$ and define $\tau = \inf\{n \geq 0 : X(n) = y\}$
- 5 For \mathcal{A} to be positive recurrent, $\mathbb{E}[\tau] < \infty$ so:

$$g(x) = \mathbb{E}[Z(0)] = \mathbb{E}[Z(\tau)] = g(y),$$

a contradiction.

Idea generalises to other structures also,



YN, Leonardo Rojas–Nandayapa and Tom Salisbury, “*Non-Existence of Stabilizing Policies for the Critical Push-Pull Network and Generalizations*”, *Operations Research Letters*, 41.3 (2013): 265-270.

Fluid intuition (Lefeber)

$$\frac{d\bar{Q}(t)}{dt} = a^* - R^* u(t)$$

with,

$$a^* := P'_{a,s} \text{diag}(\lambda) \mathbf{1}$$

$$R^* := (I - P'_{s,s}) \text{diag}(\mu) + P'_{a,s} \text{diag}(\lambda) C_s$$

If R^* is singular with a column-space containing a^*
then the system is not controllable

$$\frac{d}{dt} \begin{bmatrix} \bar{Q}_1(t) \\ \bar{Q}_2(t) \end{bmatrix} = \begin{bmatrix} \lambda_1 \\ \lambda_2 \end{bmatrix} - \begin{bmatrix} \mu_1 & \lambda_1 \\ \lambda_2 & \mu_2 \end{bmatrix} \begin{bmatrix} u_1(t) \\ u_2(t) \end{bmatrix}$$

Assume:

- Processing times with bounded residual means:

$$\mathbb{E}[U - t \mid U > t] \leq \text{const}$$

- At most one infinite supply per server: $|\mathcal{C}(i) \cap \mathcal{K}_a| \leq 1$
- Solution to static planning (u^*, v^*) yields $\tilde{\rho} < \mathbf{1}$

Then if $R^* = (I - P'_{s,s}) \text{diag}(\mu) + P'_{a,s} \text{diag}(\lambda) C_s$ is **singular** then there does **not exist** a policy achieving rates,

$$T_k(t) \sim u_k^* t, \quad k \in \mathcal{K}_s, \quad \text{and} \quad T_k(t) \sim v_k^* t, \quad k \in \mathcal{K}_a$$

that is **stable**

Proof idea

- Algebraic argument constructing a suitable linear “test-function” $g(x) = \sum_{k \in \mathcal{K}_s} h_k x_k$ based on

$$R^* = (I - P'_{s,s}) \text{diag}(\mu) + P'_{a,s} \text{diag}(\lambda) C_s,$$

when it is singular

- Then, $Z(t) := g(Q(t))$ is a martingale in the exponential processing times case and in the general case is shown to satisfy (under the stability assumption),

$$\left| \mathbb{E}[Z(\tau)] - Z(0) \right| \leq \text{const} \sum_k \mathbb{E}[S_k(T_k(\tau)) - \lambda_k T_k(\tau)] \leq \text{const}$$

with τ the hitting time of \mathcal{A} .

Thus $|Z(0)| \leq \text{const}$ and this is a contradiction.