Exploring Model Predictive Control for Queueing Networks

Johan van Leeuwaarden[a,b], Erjen Lefeber[c,*],
Yoni Nazarathy[b,c,*] and Koos Rooda[c].

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[a]Department of Mathematics and Computer Science, Eindhoven University of Technology, The Netherlands.
[b]EURANDOM, Eindhoven University of Technology, The Netherlands.
[c]Department of Mechanical Engineering, Eindhoven University of Technology, The Netherlands.

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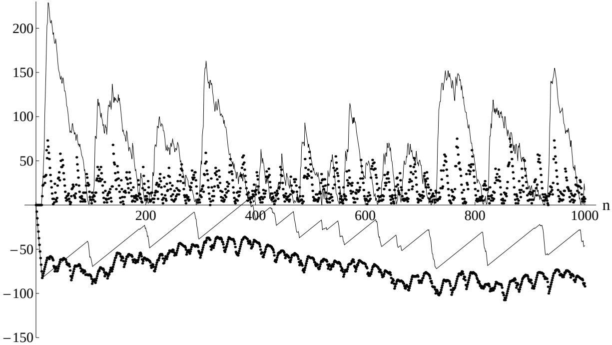
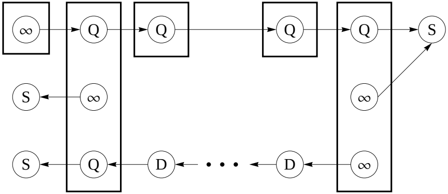
Overview and Main Story: Control of Queueing Networks

- Queueing Networks: Jobs, Servers, Queues, Routes, Scheduling Policy
- Desired:
  - High throughout, low WIP, steady output
  - Sensible computable control
  - Methodological and mathematical structure of the control
- In this talk: A control methodology based on MPC
- Not yet: Theory, stability, adaptiveness, robustness, observers...



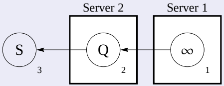
Our Queueing Network Models

- Discrete time $n = 0, 1, \ldots$
- K job classes, L servers
- Types of classes:
  - ∞ – Source
  - Q – Queue
  - D – Delay
  - S – Sink
- Deterministic routes
- Randomness due to "batch" arrivals" (∞ classes), $\tilde{u}_k$, m_k
- Processing capacity: jobs per server per time unit, $c_i, i = 1, \ldots L$
- Control Policy – How do servers allocate capacity among Q and ∞ ?

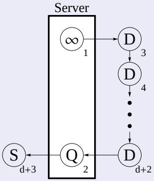


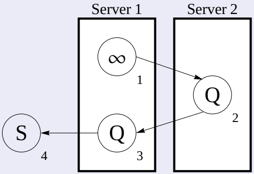
Some Structured Examples

A Single Server Queue, (c_1, c_2, m_1)



The Acquisition Queue – Similar to – A Simple Re-Entrant Line





- D. Denteneer, J. van Leeuwaarden, and I. Adan. The acquisition queue. *Queueing Systems*, 56(3):229240, 2007.
- I. Adan and G. Weiss. Analysis of a simple Markovian re-entrant line with infinite supply of work under the LBFS policy. Queueing Systems, 54(3):169183, 2006.

A Controlled Markov Chain

- $P = \{p_{kk'}\}$ – routing matrix, C – constituency matrix
- i.i.d jobs generated at sources, $\tilde{u}^{*U}$ – generic r.v. of U-fold sum
- $\{X(n)\}$ is a controlled Markov chain, with control $U(n) = f(X(n))$

$$X_k(n+1) = \begin{cases} X_k(n) + \sum_{k' \in \mathcal{K}_D} X_{k'}(n) p_{k'k} + \sum_{k' \in \mathcal{K}_{\{Q,\infty\}}} \tilde{u}_{k'}^{*U_{k'}(n)} p_{k'k} - U_k(n), & k \in \mathcal{K}_Q \text{ (queue)} \\ \sum_{k' \in \mathcal{K}_D} X_{k'}(n) p_{k'k} + \sum_{k' \in \mathcal{K}_{\{Q,\infty\}}} \tilde{u}_{k'}^{*U_{k'}(n)} p_{k'k}, & k \in \mathcal{K}_D \text{ (delay)} \\ X_k(n) + \sum_{k' \in \mathcal{K}_D} X_{k'}(n) p_{k'k} + \sum_{k' \in \mathcal{K}_{\{Q,\infty\}}} \tilde{u}_{k'}^{*U_{k'}(n)} p_{k'k}, & k \in \mathcal{K}_S \text{ (sink)} \end{cases}$$

Matrix form

$$\begin{bmatrix} X_Q(n+1) \\ X_D(n+1) \\ X_S(n+1) \end{bmatrix} = \begin{bmatrix} I & P'_{DQ} & 0 \\ 0 & P'_{DD} & 0 \\ 0 & P'_{DS} & I \end{bmatrix} \begin{bmatrix} X_Q(n) \\ X_D(n) \\ X_S(n) \end{bmatrix} + \begin{bmatrix} P'_{\infty Q} M_\infty & P'_{QQ} - I \\ P'_{\infty D} M_\infty & P'_{QD} \\ P'_{\infty S} M_\infty & P'_{QS} \end{bmatrix} \begin{bmatrix} U_\infty(n) \\ U_Q(n) \end{bmatrix} + \begin{bmatrix} P'_{\infty Q} \\ P'_{\infty D} \\ P'_{\infty S} \end{bmatrix} \tilde{u}(U_\infty(n))$$

Elements of $\tilde{u}(\cdot)$ are $\tilde{u}_k^{*U_k(n)} - U_k(n) m_k$

s.t.

$$\begin{bmatrix} 0 & 0 & 0 & -I & 0 \\ 0 & 0 & 0 & 0 & -I \\ -I & 0 & 0 & 0 & I \\ 0 & 0 & 0 & C_\infty & C_Q \end{bmatrix} \begin{bmatrix} X_Q(n) \\ X_D(n) \\ X_S(n) \\ U_\infty(n) \\ U_Q(n) \end{bmatrix} \leq \begin{bmatrix} 0 \\ 0 \\ 0 \\ c \end{bmatrix}$$

Control as a Linear System

$$X(n+1) = A\,X(n) + B\,U(n) + \text{zero mean noise}$$

s.t.

$$F \left[\begin{array}{c} X(n) \\ U(n) \end{array} \right] \leq g$$

Our Control Methodology

- Ignore noise
- Assume state and control $(X(\cdot), U(\cdot))$ are continuous in value
- Find a **reference trajectory**
- Apply "standard" control-theoretic methods for **tracking** the reference trajectory
- Use **Model Predictive Control** (MPC) using a Quadratic Programming (QP) formulation

Illustrative Example: Acquisition Queue with $d = 3$

$$\begin{bmatrix} D_1(n+1) \\ D_2(n+1) \\ D_3(n+1) \\ Q(n+1) \\ S(n+1) \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} D_1(n) \\ D_2(n) \\ D_3(n) \\ Q(n) \\ S(n) \end{bmatrix} + \begin{bmatrix} m & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & -1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} U_\infty(n) \\ U_Q(n) \end{bmatrix}$$

Maximal Throughput

$$\delta = \lim_{n\to\infty} \frac{1}{n} S(n) = m \frac{c}{1+m},$$

A Reference Trajectory

$$D_1^r(n) = D_2^r(n) = D_3^r(n) = Q^r(n) = U_\infty^r(n) = U_Q^r(n) = \delta, \quad S^r(n) = \delta\, n$$

Error Dynamics: $X^e(n) = X(n) - X^r(n)$, $U^e(n) = U(n) - U^r(n)$

Also satisfy $X^e(n+1) = A\,X^e(n) + B\,U^e(n)$

Our controller needs tries to regulate $X^e(n)$ on 0

The MPC Approach

Action of Controller at Time n

- Look at $X^e(n)$
- Plan an optimal schedule for a **time horizon** of N time units:
  - Optimize the variables $U^e(n), \ldots, U^e(n+N-1)$
  - These yield predictions of $X^e(n+1), \ldots, X^e(n+N)$
  - Practical objective (QP):

$$\sum_{i=n}^{n+N-1} \hat{X}^e(i+1)' Q \hat{X}^e(i+1) + U^e(i)' R U^e(i)$$

- After optimizing – **use first step**:
  - $U(n) = U^e(n) + U^r(n)$
  - Round off $U(n)$ and insure feasibility
- Repeat in next time step

Parameters: Time horizon, N. Positive definite cost matrixes, Q, R

Precise Formulation of the QP (for illustration)

$$\min_{\underline{U}^e} \quad \underline{U}^{e'} (\underline{B}' \, \underline{Q} \, \underline{B} + \underline{R}) \underline{U}^e + 2 X_0^{e'} \, \underline{A}' \, \underline{Q} \, \underline{B} \, \underline{U}^e$$

s.t.

$$\begin{bmatrix} \underline{C} \\ \underline{S}^1_{UQ} \\ \underline{S}^+_{UQ} - \underline{S}^-_{XQ} \underline{B} \\ -I \end{bmatrix} \underline{U}^e \leq \begin{bmatrix} \underline{c} \\ 0 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ \underline{S}^1_{XQ} \\ \underline{S}^-_{XQ} \\ 0 \end{bmatrix} \underline{X}^r + \begin{bmatrix} -\underline{C} \\ -\underline{S}^1_{UQ} \\ -\underline{S}^+_{UQ} \\ I \end{bmatrix} \underline{U}^r + \begin{bmatrix} 0 \\ S_{XQ} \\ \underline{S}^-_{XQ} \underline{A} \\ 0 \end{bmatrix} X_0^e$$

$\underline{Q}$, $\underline{R}$ are block diagonal matrixes of Q and R. The S matrixes "select" elements. The following matrixes are used for prediction:

$$\underline{A} = \begin{bmatrix} A \\ A^2 \\ \vdots \\ A^N \end{bmatrix}, \quad \underline{B} = \begin{bmatrix} B & 0 & \cdots & 0 \\ AB & B & & \vdots \\ \vdots & & \ddots & \\ A^{N-1}B & & \cdots & B \end{bmatrix}$$

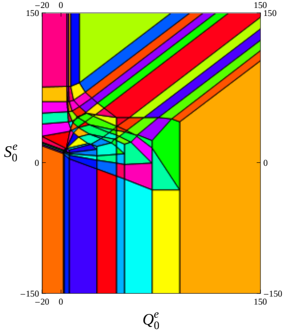
Observe: If U^r is constant as well as X^r on the Q-classes then control law is a function of X_0^e only

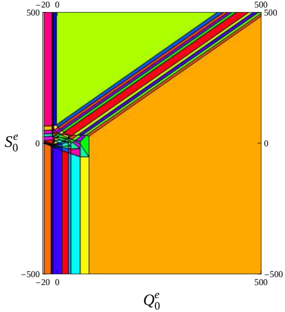
Possibility for "Explicit Form" the Solution

Multi-Parametric Quadratic Programming (MPQP)

Algorithms for an "explicit solution" in terms of X_0^e:
A piece-wise affine function

Example: Single Server Queue with $c_1 = \infty$, $c_2 = 20$, $m_1 = 1$. $Q = I$, $R = I$, $N = 5$.





M. Kvasnica, P. Grieder, M. Baotic, and M. Morari. Multi-parametric toolbox (MPT). *Hybrid Systems: Computation and Control*, pages 121124, 2004.

Stability?

Continuous Deterministic Case

- Add **end point constraint**: $X^e(N) = 0$
- Main Theoretical Result: If feasible solution exists then resulting system is asymptotically stable
- Alternative: Take $N = \infty$

Discrete Stochastic Case

- No general result
- Some hope of proving positive recurrence for "toy examples" by analyzing the solution of the QP when X_0^e is far from the origin
- Practical alternative: Use end point constraint. When QP is not feasible, don't work on ∞ classes

Numerical Examples

Acquisition Queue Threshold vs. MPC

Example: $c = 10$, $d = 10$, $m = 3$

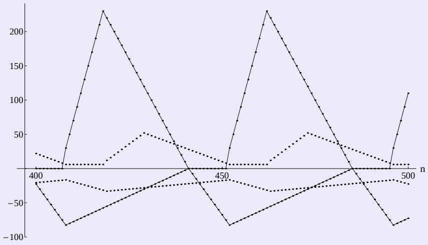
A Simple Threshold Control Law (Van Leeuwaarden et. al. 2007)

$$U_\infty(n) = \alpha + (c - Q(n))^+, \quad U_Q(n) = c - U_\infty(n)$$

$\alpha < c/(1+m)$ (for stability)

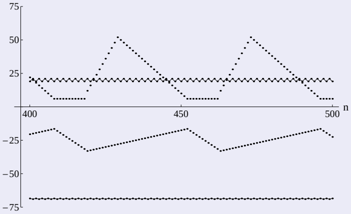
Example: $d = 10$, $m = 3$, no noise

Assume no noise, optimize α: $\alpha = 2$ is best

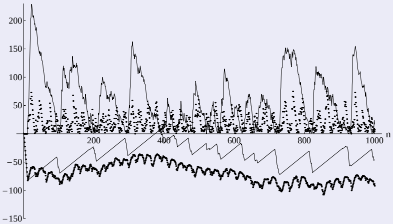


MPC Appears Better than Threshold

Compare $\alpha = 2$ with MPC ($Q = I$, $R = I$, $N = 30$)



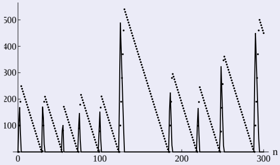
Add noise (Geometrically distributed acquisitions)

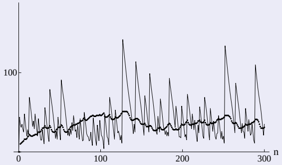


Simple Re-Entrant Line ($\rho = 1$)

$c_1 = 10,\ c_2 = 100,\ m_1 = 20,\ \delta = m_1 \frac{c_1}{1+m_1} \approx 9.52$

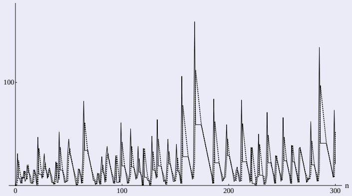
Comparing LBFS and MPC $Q = \mathbf{1}$, $R = I$, $N = 10$





Note: Both controllers achieve desired throughput

Different Objective, $Q = I$



Note: MPC does NOT achieve desired throughput

A More Complex Network

Use Linear Program (LP) to Find a Reference Trajectory

$$\max \quad \sum_{i=1}^{4} w_i r_i$$

s.t.

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 0 & 1/m_3 & 1 \\ 1 & 1/m_2 & 0 & 1/m_4 \end{bmatrix} \begin{bmatrix} r_1 \\ r_2 \\ r_3 \\ r_4 \end{bmatrix} \leq \begin{bmatrix} c_1 m_1 \wedge c_3 \wedge c_4 \\ c_2 \\ c_5 \end{bmatrix},$$

$$r_i \geq 0, \quad i = 1, 2, 3, 4.$$

$\rho \in [0, 1]$

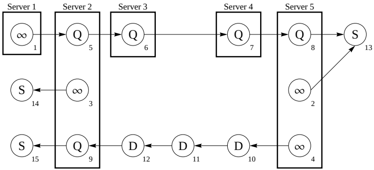
(route 1) $U_1^r(n) = \rho r_1^* / m_1, \quad X_5^r(n) = X_6^r(n) = X_7^r(n) = X_8^r(n) = U_5^r(n) = U_6^r(n) = U_7^r(n) = U_8^r(n) = \rho r_1^*,$

(route 2) $U_2^r(n) = \rho r_2^* / m_2,$

(route 3) $U_3^r(n) = \rho r_3^* / m_3,$

(route 4) $U_4^r(n) = \rho r_4^* / m_4, \quad X_{10}^r(n) = X_{11}^r(n) = X_{12}^r(n) = X_9^r(n) = U_9^r(n) = \rho r_4^*,$

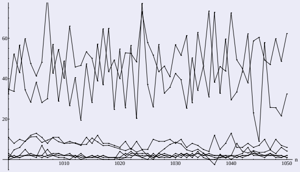
(sinks) $X_{13}^r(n) = \rho(r_1^* + r_2^*)n, \quad X_{14}^r(n) = \rho r_3^* n, \quad X_{15}^r(n) = \rho r_4^* n.$



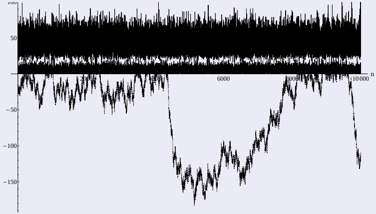
A More Complex Network (cont.)

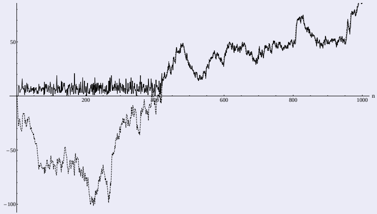
Parameters set such that $r_i^* > 0$, $i = 1, 2, 3, 4$, $Q = I$, $R = I$, $\rho = 1$

$N = 5$, Stable



$N = 4$, Stable in Q's (not in output). $N = 3$ Unstable





Conclusion

- Main idea: View queueing network as controlled linear system with noise – apply MPC – appears to "work well"
- General theory: (1) Stability properties (2) Bounds on performance... "hard to obtain"
- At least... Hope for explicit stochastic analysis of some toy examples
- Immediate extensions (in progress): (1) Observers (2) inverse optimality of other controllers - e.g. dead beat
- Interesting to experiment: Incorporating the effect of noise in the reference value for the queue levels

Questions?