Positive Harris Recurrence and Diffusion Scale Analysis of a Push Pull Queueing Network

Yoni Nazarathy   Gideon Weiss

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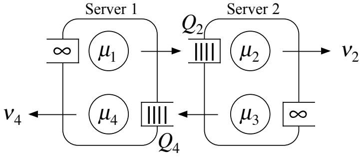
Outline

Outline

Preview of Results – The Push Pull Network



- *THEOREM 1*: Fluid stability.
- *THEOREM 2*: Positive Harris recurrence.
- *THEOREM 3*: Diffusion approximation.

Insights

- Full Utilization and stable network – general processing times.
- Diffusion scale covariance between outputs – negative.
- Diffusions of outputs – same for all stable policies.

Outline

Outline

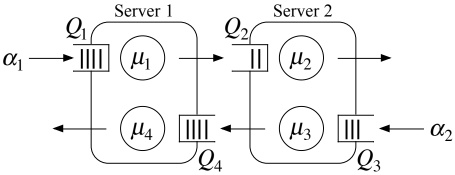
Single class networks

- Jackson Networks.
- Generalized Jackson Networks.

Multi-class networks – several job classes per node

- Choose a policy.
- Analysis is hard, policy dependent.
- Optimal exact solutions – even harder.
- Optimal approximate solutions – sometimes possible.

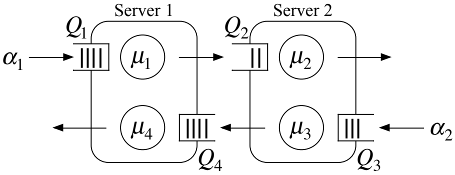
Stability Depends on Policy – KSRS Example



Kumar-Seidman, Rybko-Stoylar (90's)

- Offered loads: $\rho_1 = \frac{\alpha_1}{\mu_1} + \frac{\alpha_2}{\mu_4}, \qquad \rho_2 = \frac{\alpha_1}{\mu_2} + \frac{\alpha_2}{\mu_3}$
- Necessary Condition for stability: $\rho_1, \rho_2 < 1$.
- Condition is not sufficient – KSRS example.

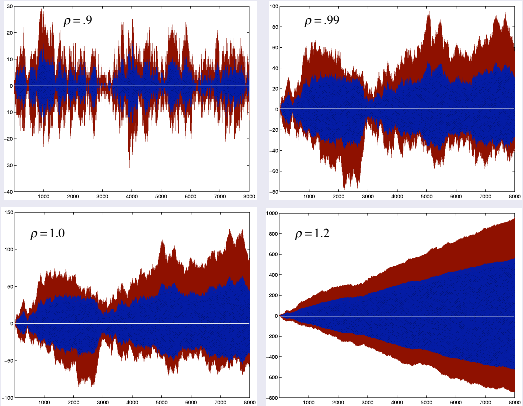
KSRS in Balanced Heavy Traffic



- Some "good" policies exist (e.g Max-Pressure [Dai & Lin]).
- Yet as $\alpha_i \nearrow \implies \rho_i \to 1 \implies$ congestion.

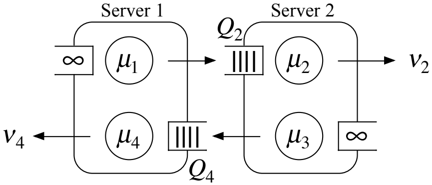
KSRS in Balanced Heavy Traffic

Max-Pressure, $\mu_1 = \mu_3 = 1.25 \quad \mu_2 = \mu_4 = 1 \quad \alpha_i \nearrow$



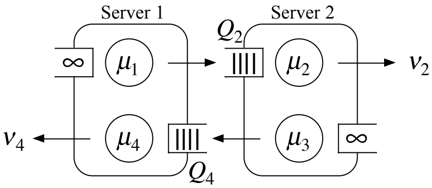
Outline

The Push Pull Network



- No input stream –infinite supply of work.
- Steps 1 and 3 push, Steps 2 and 4 pull.
- Infinite supply allows full utilization.
- Two queues $Q_2(t)$, $Q_4(t)$, no congestion.

Rates for Full Utilization



- θ_i – allocation to i.
- Require: Q_2, Q_4 not congested.

$$\theta_1 = 1 - \theta_4,\ \theta_2 = 1 - \theta_3 \qquad \mu_1\theta_1 = \mu_2\theta_2,\ \mu_3\theta_3 = \mu_4\theta_4$$

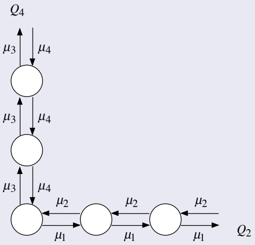
$$\nu_1 = \nu_2 = \frac{\mu_1\mu_2(\mu_3 - \mu_4)}{\mu_1\mu_3 - \mu_2\mu_4}, \quad \nu_3 = \nu_4 = \frac{\mu_3\mu_4(\mu_1 - \mu_2)}{\mu_1\mu_3 - \mu_2\mu_4}.$$

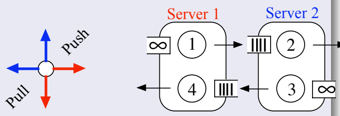
Case 1 – Inherently Stable

Case 1

- Parameters: $\mu_1 < \mu_2, \ \mu_3 < \mu_4$.
- Policy: Pull Priority.

Exponential Processing Times – Kopzon & Weiss (2001)



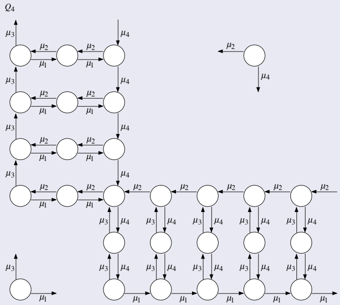


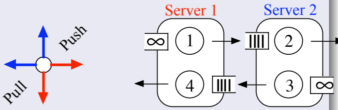
Case 2 – Inherently Unstable

Case 2

- Parameters: $\mu_1 > \mu_2,\ \mu_3 > \mu_4$.
- Policy: Threshold – push when other server's queue low.

Exponential Processing Times – Kopzon & Weiss (2007)





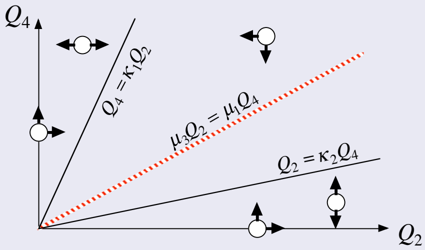
For Case 2, Use Linear Threshold Policy

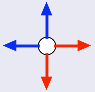
Why not fixed thresholds?

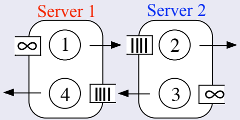
- Threshold is distribution dependent: $\left(\frac{\mu_1}{\mu_2}\right)^{s_2}\left(\frac{\mu_4}{\mu_3}\right) > 1$, $\left(\frac{\mu_3}{\mu_4}\right)^{s_4}\left(\frac{\mu_2}{\mu_1}\right) > 1$.
- Fluid approximation – how?

Linear thresholds

Choose $\kappa_1 > \frac{\mu_3}{\mu_1}$, $\kappa_2 > \frac{\mu_1}{\mu_3}$,

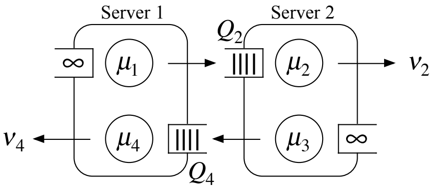






Outline

Outline of Analysis and Results



THEOREM 1: Fluid stability.

THEOREM 2: Positive Harris recurrence.

THEOREM 3: Diffusion approximation.

Outline

A Multi-Class Queueing Network Model

Random processing time sequences

$\xi_i = \{\xi_i^j,\ j = 1, 2, \dots\}$, $\quad S_i(t) =$ departure in processing t.

Allocation processes: $T_i(t)$

$T_i(0) = 0$, $\quad T_i(\cdot) \nearrow$, $\quad T_i(t) - T_i(s) \le t - s$

Lipschitz, absolutely continuous, derivative exists a.e.

Outputs and queue lengths

Outputs: $D_i(t) = S_i(T_i(t))$. Queue Lengths: $Q_2(t), Q_4(t) \ge 0$.

$$Q_2(t) = Q_2(0) + D_1(t) - D_2(t), \quad Q_4(t) = Q_4(0) + D_3(t) - D_4(t)$$

Policies

- Head of the line.
- Preemptive resume.

Servers never idle

$$T_1(t) + T_4(t) = t, \qquad T_2(t) + T_3(t) = t.$$

Case 1 – Pull priority

$$\int_0^t Q_4(s)dT_1(s) = 0, \qquad \int_0^t Q_2(s)dT_3(s) = 0.$$

Case 2 – Linear thresholds

$$\begin{aligned} \int_0^t \mathbf{1}\{0 < Q_4(s) < \kappa_1 Q_2(s)\}dT_1(s) &= 0 \\ \int_0^t \mathbf{1}\{0 \le Q_2(s) \le \tfrac{1}{\kappa_1} Q_4(s)\}dT_4(s) &= 0 \end{aligned} \qquad \begin{aligned} \int_0^t \mathbf{1}\{0 \le Q_4(s) \le \tfrac{1}{\kappa_2} Q_2(s)\}dT_2(s) &= 0 \\ \int_0^t \mathbf{1}\{0 < Q_2(s) < \kappa_2 Q_4(s)\}dT_3(s) &= 0 \end{aligned}$$

Assumptions

A1 – Rates (fluid scaling)

$$\lim_{n\to\infty} \frac{\sum_{j=1}^{n} \xi_i^j}{n} = \frac{1}{\mu_i} \text{ a.s.}$$

A2 – Renewal processing (positive Harris recurrence)

- (a) ξ_i, $i = 1, 2, 3, 4$, are mutually independent i.i.d.
- Technical Assumptions: (b) Spread-out and unbounded ξ_i for $i = 1, 3$, or (b') Compacts are petite.

A3 – Second moments (diffusion scaling)

$$\mu_i^2 \text{Var}(\xi_i^1) = c_i^2 < \infty.$$

Outline

Fluid Scaled Processes

Assume rates exist (A1)

$$\lim_{n\to\infty} \frac{\sum_{j=1}^{n} \xi_i^j}{n} = \frac{1}{\mu_i} \quad \text{a.s.}$$

A sequence of processes parameterized $n = 1, 2, \ldots$

- $Y^n(t) = (Q^n(t), T^n(t))$ all sharing the same $\xi_i(\omega)$.
- $Q^n(0)$: A sequence of initial queue lengths.
- Fluid Scalings: $\bar{Y}^n(t,\omega) = \frac{Y^n(nt,\omega)}{n}$.

Definition

$\bar{Y}(t) = (\bar{Q}(t), \bar{T}(t))$ is a fluid limit of the network:
for some sample path ω and some $r \to \infty$, $\bar{Y}^r(\cdot,\omega) \to \bar{Y}(\cdot)$, u.o.c.

Fluid Limit Model and Fluid Model Equations

Theorem

Fluid limits exist. They form the fluid limit model. They satisfy:

Fluid Model Equations

$$\bar{Q}_i(t) = \bar{Q}_i(0) + \mu_{i-1}\,\bar{T}_{i-1}(t) - \mu_i\,\bar{T}_i(t) \geq 0\,, i = 2, 4.$$
$$\bar{T}_i(0) = 0,\ \ \bar{T}_i\ \nearrow,\ \ \bar{T}_i(t) - \bar{T}_i(s) \leq t - s,\ \ \ i = 1, 2, 3, 4.$$

and additional equations which depend on the policy.

Policy Related Fluid Model Equations

Full utilization

$$\bar{T}_1(t) + \bar{T}_4(t) = t, \qquad \bar{T}_2(t) + \bar{T}_3(t) = t.$$

Case 1 – Pull priority policy

$$\int_0^t \bar{Q}_4(s)d\bar{T}_1(s) = 0, \qquad \int_0^t \bar{Q}_2(s)d\bar{T}_3(s) = 0.$$

Case 2 – Linear threshold policy

$$\begin{aligned} \int_0^t \mathbf{1}\{0 < \bar{Q}_4(s) < \kappa_1 \bar{Q}_2(s)\} d\bar{T}_1(s) &= 0 \\ \int_0^t \mathbf{1}\{0 \leq \bar{Q}_2(s) \leq \tfrac{1}{\kappa_1} \bar{Q}_4(s)\} d\bar{T}_4(s) &= 0 \end{aligned} \qquad \begin{aligned} \int_0^t \mathbf{1}\{0 \leq \bar{Q}_4(s) \leq \tfrac{1}{\kappa_2} \bar{Q}_2(s)\} d\bar{T}_2(s) &= 0 \\ \int_0^t \mathbf{1}\{0 < \bar{Q}_2(s) < \kappa_2 \bar{Q}_4(s)\} d\bar{T}_3(s) &= 0 \end{aligned}$$

Fluid Stability Result

Definitions

- $(\bar{Q}, \bar{T})$ that satisfies the fluid model equations is a fluid solution.
- A fluid model is stable if there exists a $\delta > 0$ such that for every fluid solution with $|\bar{Q}(0)| = 1$, $Q(t) = 0$ for any $t \geq \delta$.

THEOREM 1 Fluid Stability of Push Pull Network

The fluid model of the push pull network, for Case 1 with pull priority and for Case 2 with linear threshold policy, is stable.

Corollary Fluid Approximation

For fixed $Q(0)$, $Y(nt)/n$ converges as $n \to \infty$ u.o.c. a.s. to a fluid limit $\bar{Y}(t)$ with: $\bar{T}_i(t) = \theta_i t, \quad \bar{D}_i(t) = \nu_i t, \quad \bar{Q}_i(t) = 0$.

Lyapounov Proof for Fluid Stability

Lemma

Let f be absolutely continuous, $f(t) \geq 0$, with derivative $\dot{f}(t)$ at regular points. Assume:

(i) f(t) = 0 implies $\dot{f}(t) = 0$.

(ii) There exists $\epsilon > 0$ for which at regular t: If $f(t) > 0$ then $\dot{f}(t) \leq -\epsilon$.

Then: $f(t) = 0$ for all $t > f(0)/\epsilon$, furthermore $f(\cdot)$ is non-increasing so once it reaches 0 it stays there.

Lyapunov Function

$g(\bar{Q}_2, \bar{Q}_4) \geq 0$, and $g(\bar{Q}_2, \bar{Q}_4) = 0$ if and only if $\bar{Q}_2 = \bar{Q}_4 = 0$.
$|\bar{Q}| = \bar{Q}_2 + \bar{Q}_4 = 1$ then $g(\bar{Q}_2, \bar{Q}_4) \leq B$.
Let $f(t) = g(\bar{Q}_2(t), \bar{Q}_4(t))$, if f satisfies the assumptions of the Lemma, then the network fluid model is stable, with $\delta = B/\epsilon$.

Lyapounov function $f(t) = \bar{Q}_2(t) + \bar{Q}_4(t)$.
Note: $f(t) \geq 0$ and $f(t) = 0$ if and only if $\bar{Q}(t) = 0$ and $|\bar{Q}(0)| = 1$ then $f(0)$ is bounded.
Take $\epsilon = \min\{\mu_2 - \mu_1, \mu_4 - \mu_3$. Now bound $\dot{f}(t)$:

- $\bar{Q}_2(t),\ \bar{Q}_4(t) > 0$:
  $\dot{\bar{T}}_1 = \dot{\bar{T}}_3 = 0, \quad \dot{\bar{T}}_2 = \dot{\bar{T}}_4 = 1, \quad \dot{\bar{Q}}_i(t) = -\mu_i$
  $\dot{f}(t) = -(\mu_2 + \mu_4)$.
- $\bar{Q}_2(t) > 0$, $\bar{Q}_4(t) = 0$:
  $\dot{\bar{T}}_3 = 0, \quad \dot{\bar{T}}_2 = 1,$

$$\begin{aligned} \dot{f}(t) &= \mu_1 \dot{\bar{T}}_1(t) - \mu_2 - \mu_4 \dot{\bar{T}}_4(t) = \\ &= \mu_1 - \mu_2 - (\mu_1 + \mu_4)\dot{\bar{T}}_4(t) \leq -(\mu_2 - \mu_1) \end{aligned}$$

- $\bar{Q}_2(t) = 0$, $\bar{Q}_4(t) > 0$, Similarly,
  $\dot{f}(t) \leq -(\mu_4 - \mu_3)$.

Lyapounov Proof for Fluid Stability

Case 2 – Same Technique – Different Lyapounov Function

Piecewise linear Lyapounov function

$$f(t) = \begin{cases} (1+\beta)\bar{Q}_2(t) - (\kappa_2 - \beta)\bar{Q}_4(t) & \text{if } \bar{Q}_2(t) \geq \kappa_2 \bar{Q}_4(t), \\ (1+\beta)\bar{Q}_4(t) - (\kappa_1 - \beta)\bar{Q}_2(t) & \text{if } \bar{Q}_4(t) \geq \kappa_1 \bar{Q}_2(t), \\ \beta(\bar{Q}_2(t) + \bar{Q}_4(t)) & \text{otherwise.} \end{cases}$$

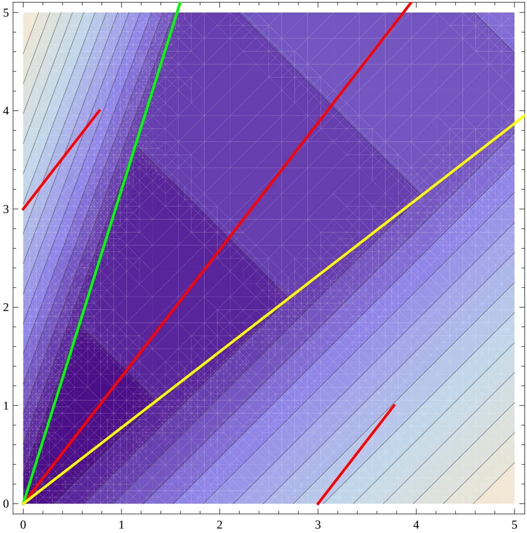
where

$$\beta = \frac{1}{2}\min\{\frac{\kappa_1 - \frac{\mu_3}{\mu_1}}{1 + \frac{\mu_3}{\mu_1}}, \frac{\kappa_2 - \frac{\mu_1}{\mu_3}}{1 + \frac{\mu_1}{\mu_3}}\}.$$

We again obtain $\dot{f} \leq -\epsilon$ and $f(t) \geq 0$ and $f(t) = 0$ if and only if $Q(t) = 0$, and if $|\bar{Q}(0)| = 1$ then $f(0)$ is bounded by some finite value B.

Lyapounov Proof for Fluid Stability

Case 2 – same technique – different Lyapounov function



Outline

Markovian Network State Process

Assume Renewal processing (A2)

$$\xi_i,\ i = 1, 2, 3, 4 \text{ are mutually independent i.i.d.}$$

A Markovian State

- Define the *network state process*, $X(t) = (Q(t), U(t))$. $(U_1(t), U_3(t)), (U_2(t), U_4(t))$ residual processing times.
- State space: $\mathbb{S} = \mathbb{Z}_+^2 \times \mathbb{R}_+^2 \times \mathbb{R}_+^2$.
- Evolution of $X(t)$ between arrivals and departures is deterministic, $X(t)$ is piecewise deterministic.

Proposition

$X = \{X(t),\ t \geq 0\}$ is a strong Markov process on $\mathbb{S}$.

Positive Harris Recurrence (PHR)

- For $x \in \mathbb{S}$, $B \in \mathcal{B}(\mathbb{S})$
  $P^t(x,B) = P_x(X(t) \in B) = P\{X(t) \in B | X(0) = x\}$.
- π σ-finite on $(\mathbb{S}, B(\mathbb{S}))$ is invariant if for all t

$$\pi(B) = \int_{\mathbb{S}} P^t(x,B)\,\pi(dx), \quad B \in \mathcal{B}(\mathbb{S}).$$

- $\tau_A = \inf\{t \geq 0 : X(t) \in A\}$. X is Harris recurrent if for measure ν, $A \in \mathcal{B}(\mathbb{S})$ with $\nu(A) > 0$ implies $P_x(\tau_A < \infty) = 1$ for all $x \in \mathbb{S}$.
- Harris Recurrent $\Longrightarrow$ $\exists$ invariant measure π.
- When π is a probability X is positive Harris recurrent.
- PHR implies ergodicity: for all $x \in \mathbb{S}$, and all $f \geq 0$

$$\lim_{t\to\infty} \frac{1}{t} \int_0^t f(X(s))ds = \int_{\mathbb{S}} f(x)\,\pi(dx) \qquad P_x \text{ a.s.}$$

Main Stability Result – PHR

THEOREM 2: Positive Harris Recurrence

Under Assumptions (A1), (A2a) and (A2b'), the network state process X is PHR for Case 1 under the pull priority policy and for Case 2 under the linear threshold policy. Furthermore, for Case 1 we may substitute Assumptions (A2b') with (A2b).

Proof

We use the framework of Dai '95. Dai has shown that for a MCQN (with external arrival streams), fluid model stability implies PHR. With infinite virtual queues we need to modify the proof.

Dai's Framework

Dai shows that PHR follows from two statements:

(i) Convergence of process scaled by initial state

$$\lim_{|x|\to\infty} \frac{1}{|x|} E_x \left| X(\delta|x|) \right| = 0, \text{some} \delta > 0$$

(ii) Every compact set is petite (A2b')

The argument that (i) and (ii) imply PHR needs no modification.

Dai's main result is that fluid model stability implies (i). This also needs no modification.

Hence, by our Theorem on Fluid Stability, under (A2a,A2b'), PHR follows.

Petite Sets and Compact Sets

$A \in \mathbb{S}$ is petite if there exists a probability distribution **a** on $(0, \infty)$ and a nontrivial measure ν on $(\mathbb{S}, \mathcal{B}(\mathbb{S}))$, such that for all $x \in A$

$$\int_0^\infty P^t(x, B)\mathbf{a}(dt) \geq \nu(B), \qquad \text{for all } B \in \mathcal{B}(\mathbb{S}).$$

Petiteness of A may be interpreted as the property that all sets B are "equally accessible" from any $x \in A$.

Compact sets in $\mathbb{S}$ are closed sets with bounded Q_i, U_i.
In fluid scaling they represent the point 0.

Petiteness is a property of the Markov Process, and therefore depends on the policy.

It is not straightforward to check if compact sets are petite.

The Technical Assumptions (A2b') and (A2b)

Our technical assumptions are

- (A2b') Every compact set is petite
- (A2b) Spread out unbounded processing times:
  For the push activities (Infinite Virtual Queues)

$$P(\xi_i^1 \geq x) > 0 \text{ for all } x > 0,$$

$$\exists k_0^i > 0,\ q_i(\cdot) \geq 0 \text{ with } \int_0^\infty q_i(x)dx > 0 :$$
$$P(\xi_i^1 + \ldots + \xi_i^{k_0^i} \in dx) \geq q_i(x)dx, \qquad i = 1, 3.$$

Proposition

Under pull priority policy, (A2b) implies (A2b')

We were unable to prove a similar result for the linear threshold policy

Outline

Diffusion Scaling Relations – Finite n

Diffusion Scalings

$$\hat{S}_i^n(t) = \frac{S_i(nt) - \bar{S}_i(nt)}{\sqrt{n}}, \qquad \hat{T}_i^n(t) = \frac{T_i(nt) - \bar{T}_i(nt)}{\sqrt{n}},$$
$$\hat{D}_i^n(t) = \frac{D_i(nt) - \bar{D}_i(nt)}{\sqrt{n}}, \qquad \hat{Q}_i^n(t) = \frac{Q_i(nt)}{\sqrt{n}}.$$

where (fluid scalings and limits),

$$\begin{array}{rcll} \bar{S}_i^n(t) & = & \frac{S_i(nt)}{n} \longrightarrow \bar{S}_i(t) = \mu_i t & \\ \bar{T}_i^n(t) & = & \frac{T_i(nt)}{n} \longrightarrow \bar{T}_i(t) = \theta_i t & , \text{ a.s. (u.o.c) } n \to \infty \\ \bar{D}_i^n(t) & = & \frac{D_i(nt)}{n} \longrightarrow \bar{D}_i(t) = \theta_i \mu_i t & \end{array}$$

Decomposition of output variability

$$\begin{aligned} \hat{D}_i^n(t) &= \frac{D_i(nt) - \bar{D}_i(nt)}{\sqrt{n}} \\ &= \frac{S_i(n\bar{T}_i^n(t)) - \bar{S}_i(n\bar{T}_i^n(t))}{\sqrt{n}} + \frac{\bar{S}_i(n\bar{T}_i^n(t))}{\sqrt{n}} - \frac{\bar{D}_i(nt)}{\sqrt{n}} \\ &= \hat{S}_i^n(\bar{T}_i^n(t)) + \mu_i \frac{T_i(nt) - \bar{T}_i(nt)}{\sqrt{n}} + \mu_i \frac{\bar{T}_i(nt)}{\sqrt{n}} - \frac{\bar{D}_i(nt)}{\sqrt{n}} \\ &= \hat{S}_i^n(\bar{T}_i^n(t)) + \mu_i \hat{T}_i^n(t) + \theta_i \mu_i \sqrt{n}t - \theta_i \mu_i \sqrt{n}t \;=\; \hat{S}_i^n(\bar{T}_i^n(t)) + \mu_i \hat{T}_i^n(t) \end{aligned}$$

Diffusion Scaling Relations – Finite n

Relations between diffusion scalings

$$\left\{ \begin{array}{l} \hat{D}_1^n(t) = \hat{S}_1^n(\bar{T}_1^n(t)) + \mu_1 \hat{T}_1^n(t) \\ \hat{D}_2^n(t) = \hat{S}_2^n(\bar{T}_2^n(t)) + \mu_2 \hat{T}_2^n(t) \\ \hat{D}_3^n(t) = \hat{S}_3^n(\bar{T}_3^n(t)) + \mu_3 \hat{T}_3^n(t) \\ \hat{D}_4^n(t) = \hat{S}_4^n(\bar{T}_4^n(t)) + \mu_4 \hat{T}_4^n(t) \end{array} \right\}, \left\{ \begin{array}{l} \hat{Q}_2^n(t) = \hat{D}_1^n(t) - \hat{D}_4^n(t) \\ \hat{Q}_4^n(t) = \hat{D}_3^n(t) - \hat{D}_4^n(t) \end{array} \right\}, \left\{ \begin{array}{l} \hat{T}_2^n(t) = -\hat{T}_3^n(t) \\ \hat{T}_4^n(t) = -\hat{T}_1^n(t) \end{array} \right\}$$

"Limit Ready" Representation

$$\begin{bmatrix} \hat{D}_2^n(t) \\ \hat{D}_4^n(t) \\ \hat{T}_2^n(t) \\ \hat{T}_4^n(t) \end{bmatrix} = \mathbf{A} \cdot \begin{bmatrix} \hat{S}_1^n(\bar{T}_1^n(t)) \\ \hat{S}_2^n(\bar{T}_2^n(t)) \\ \hat{S}_3^n(\bar{T}_3^n(t)) \\ \hat{S}_4^n(\bar{T}_4^n(t)) \end{bmatrix} + \mathbf{B} \cdot \begin{bmatrix} \hat{Q}_2^n(t) \\ \hat{Q}_4^n(t) \end{bmatrix}$$

where,

$$\mathbf{A} = \frac{1}{\mu_1\mu_3 - \mu_2\mu_4} \begin{bmatrix} -\mu_2\mu_4 & \mu_1\mu_3 & \mu_1\mu_2 & -\mu_1\mu_2 \\ \mu_3\mu_4 & -\mu_3\mu_4 & -\mu_2\mu_4 & \mu_1\mu_3 \\ -\mu_4 & \mu_4 & \mu_1 & -\mu_1 \\ \mu_3 & -\mu_3 & -\mu_2 & \mu_2 \end{bmatrix}, \quad \mathbf{B} = \frac{1}{\mu_1\mu_3 - \mu_2\mu_4} \begin{bmatrix} \mu_2\mu_4 & -\mu_1\mu_2 \\ -\mu_3\mu_4 & \mu_2\mu_4 \\ \mu_4 & -\mu_1 \\ -\mu_3 & \mu_2 \end{bmatrix}.$$

Diffusion Result

THEOREM 3: Diffusion Limit

Under assumption (A1)–(A3), as $n \to \infty$, $(\hat{D}^n(t), \hat{T}^n(t), \hat{Q}^n(t))$ converges weakly to a 10 dimensional driftless Brownian motion. Furthermore:

$$\begin{aligned} \hat{D}_1^n(t) - \hat{D}_2^n(t) = \hat{Q}_2^n(t) &\Rightarrow 0, \\ \hat{D}_4^n(t) - \hat{D}_3^n(t) = \hat{Q}_4^n(t) &\Rightarrow 0. \end{aligned}$$

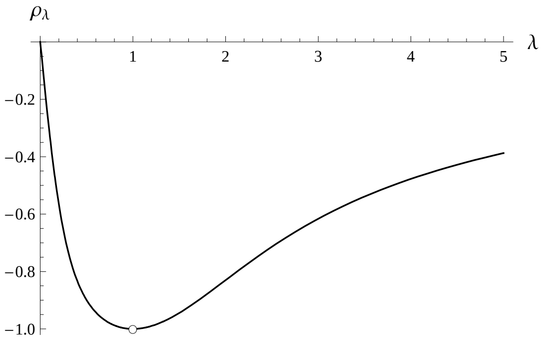
Some Elements of the Covariance Matrix

$$\begin{aligned} \mathrm{Var}(\hat{D}_2(1)) &= \frac{\mu_1\mu_2}{(\mu_1\mu_3 - \mu_2\mu_4)^3} \times \{\mu_1\mu_2\mu_3\mu_4(c_3^2 + c_4^2)(\mu_1 - \mu_2) + (\mu_1^2\mu_3^2c_2^2 + \mu_2^2\mu_4^2c_1^2)(\mu_3 - \mu_4)\} \\ \mathrm{Cov}(\hat{D}_2(1), \hat{D}_4(1)) &= -\frac{\mu_1\mu_2\mu_3\mu_4}{(\mu_1\mu_3 - \mu_2\mu_4)^3} \times \{(\mu_1\mu_3c_4^2 + \mu_2\mu_4c_3^2)(\mu_1 - \mu_2) + \mu_1\mu_3c_2^2 + \mu_2\mu_4c_1^2)(\mu_3 - \mu_4)\} \end{aligned}$$

Highly Negative Correlation Between Outputs

Covariance for symmetric push pull

Take $c_i^2 = c^2$, equal, $\mu_2 = \mu_4 = 1$, $\mu_1 = \mu_3 = \lambda$.
Plot $\rho_\lambda = \frac{\mathrm{Cov}(\hat{D}_2(1), \hat{D}_4(1))}{\sqrt{\mathrm{Var}(\hat{D}_2(1))\mathrm{Var}(\hat{D}_4(1))}}$.



Insensitivity to Specific Policy

"Inputs" to Diffusion Proof

- $\hat{Q}_i^n(t) \Rightarrow 0$ (weakly).
- $\bar{T}^n(t) \rightarrow \theta t$ (u.o.c.).
- Assumptions (A3) – Second Moment.

Conclusion

Asymptotic variance rate of outputs from fully utilizing stable push pull network does not depend on specific policy.

Outline

Open Questions and Future Work

- Petiteness of compacts – other policies.
- Anomalies in simulation and numerical approximations.
- Push pull in heavy traffic $\Rightarrow$ State Space Collapse?
- Behavior of MCQN with infinite virtual queues.

Thank You