Scaling Limits, Cyclically Varying Birth-Death Processes and Stationary Distributions

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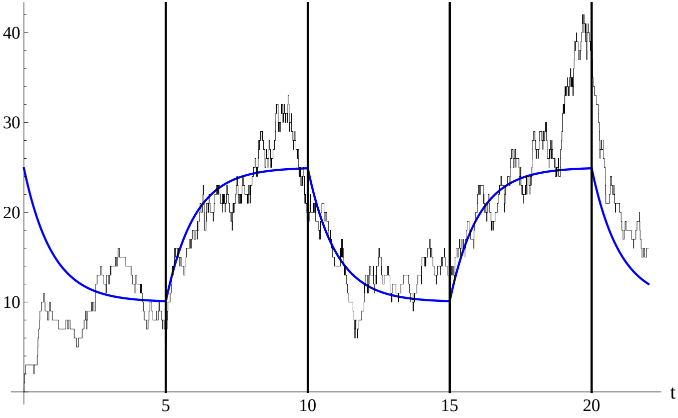
Overview

- Birth death processes
- Their scaling limits
- Cyclically varying systems



An Example Class of Birth Death Processes

- $\{X(t), t \geq 0\}$ is a Continuous Time, Birth-Death, Markov Chain taking values $\{0, 1, \ldots\}$
- Birth rates are constant: $\lambda > 0$
- Death rates are state dependent: $\mu X(t)^\alpha$
- $\alpha = 0$ is M/M/1, $\alpha = 1$ is M/M/∞

- For $\alpha > 0$, $X(t)$ is always positive recurrent
- $\pi_n = \lim_{t \to \infty} P(X(t) = n)$
- $\pi_n = \pi_0 \left(\frac{\lambda}{\mu}\right)^n \left(n!\right)^{-\alpha}$
- Closed form expressions of π_0 for α integer

Scaling the Trajectories

A related deterministic trajectory

An ODE

$$
\begin{aligned}
\dot{x}(t) &= \lambda - \mu x(t)^{\alpha} \\
x(0) &= X(0), \quad \text{(the initial value of stochastic process)} \\
m &= \lim_{t \to \infty} x(t)
\end{aligned}
$$

- For $\alpha = 1$, $m = \sum_{j=0}^{\infty} j\pi_j = \frac{\lambda}{\mu}$:

$$x(t) = m + \big(X(0) - m\big)e^{-\mu t}$$

- For $\alpha \neq 1$, $m \neq \sum_{j=0}^{\infty} j\pi_j$

Scaling The Processes

A sequence of processes

- $X_N(\cdot)$, $N = 1, 2, \ldots$
- The parameters of the N'th process: λ_N, μ_N and α
- Initial values are $X_N(0) = N\, X(0)$
- Desired: $X_N(t) \approx N\, x(t)$ as $N \to \infty$ (for finite t)

The "correct" scaling

$$\lambda_N = \lambda N, \;\; \mu_N = \mu N^{1-\alpha}$$

Why this Scaling?

Martingale Representation

$$X_N(t) = X_N(0) + M_N(t) + \lambda_N t - \mu_N \int_0^t X_N(s)^\alpha ds$$

$M_N(t)$ is a martingale

Use: $X_N(0) = N\,X(0)$, $\lambda_N = \lambda N$, $\mu_N = \mu N^{1-\alpha}$

$$\frac{X_N(t)}{N} = X(0) + \frac{M_N(t)}{N} + \lambda t - \mu \int_0^t \left(\frac{X_N(s)}{N}\right)^\alpha ds$$

Compare with

$$\dot{x}(t) = \lambda - \mu x(t)^\alpha, \qquad x(0) = X(0)$$

Theorem

(i) Trajectories:

$$\lim_{N\to\infty} P\Big(\sup_{s\in[0,t]} \Big| \frac{X_N(s)}{N} - x(s) \Big| > \epsilon \Big) = 0$$

(ii) Hitting Times:

$$\lim_{N\to\infty} P\Big(\Big| \mathcal{T}_N(yN) - \tau(y) \Big| > \epsilon \Big) = 0$$

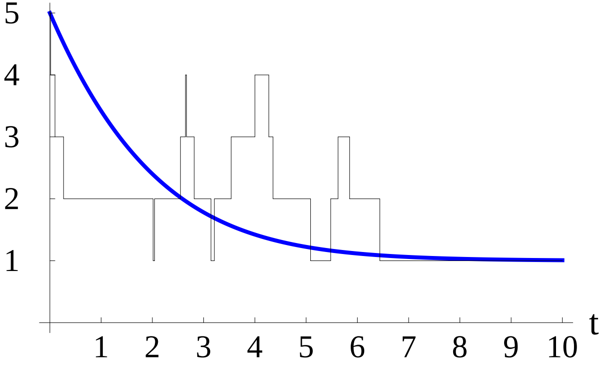
where,

$$\mathcal{T}_N(y) = \inf\{t : X_N(t) = y\}, \quad \tau(y) = \inf\{t : x(t) = y\} = x^{-1}(y)$$

Note: For $\alpha = 0, 1$ it is well known, see P. Robert book, 2003

Illustration for $\alpha = 2/3$

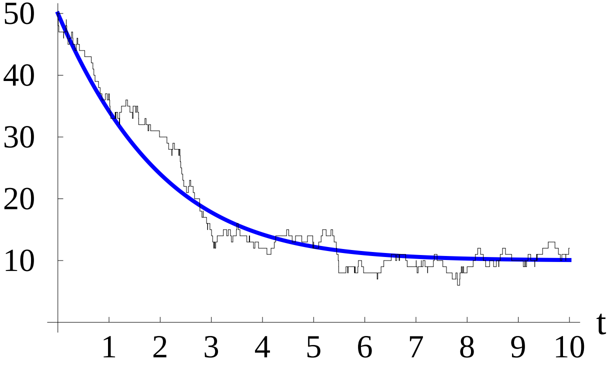
$$N = 1$$



$$\lambda = \mu = 1, \quad X(0) = 5$$

Illustration for $\alpha = 2/3$

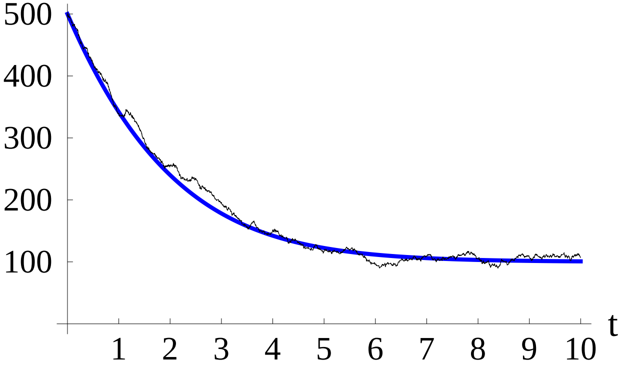
$N = 10$



$\lambda = \mu = 1, \quad X(0) = 5$

Illustration for $\alpha = 2/3$

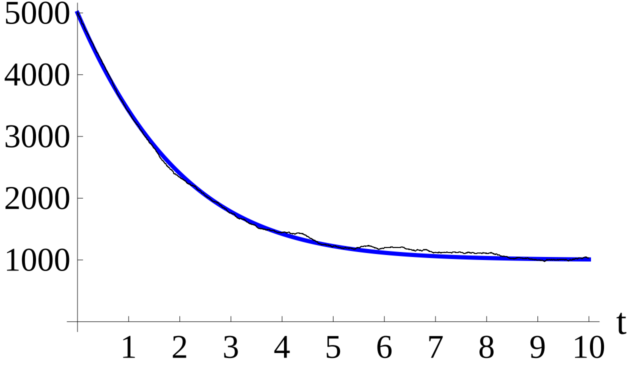
$$N = 100$$



$$\lambda = \mu = 1, \quad X(0) = 5$$

Illustration for $\alpha = 2/3$

$$N = 1000$$



$\lambda = \mu = 1, \quad X(0) = 5$

A General Formulation

- $X_N(\cdot)$ a sequence of processes with rates $\lambda_N(y), \mu_N(y)$
- $\mathcal{C}_N$, $N = 1, 2, \ldots$ a sequence of subsets of the state space
- $x(\cdot)$, solution of $\dot{x}(t) = b\big(x(t)\big) - d\big(x(t)\big)$, $x(0) = X(0)$

Notation: $\bar{g}(\mathcal{C}_N) = \sup_{y \in \mathcal{C}_N} g(y)$, for a function $g(\cdot)$

Theorem

Assume: (i) $\exists N_0 : \forall N \geq N_0,\ \lfloor N\, x(t) \rfloor \in \mathcal{C}_N$

(ii) $\bar{\lambda}_N(\mathcal{C}_N) = o(N^2)$, $\bar{\mu}_N(\mathcal{C}_N) = o(N^2)$

(iii) $\exists L, \forall N, \forall y \in \mathcal{C}_N, \forall y'$:

$$\left| \frac{\lambda_N(y)}{N} - b(y') \right| \leq L \left| \frac{y}{N} - y' \right|, \quad \left| \frac{\mu_N(y)}{N} - d(y') \right| \leq L \left| \frac{y}{N} - y' \right|$$

Then,

$$\lim_{N \to \infty} P\left(\sup_{s \in [0,t]} \left| \frac{X_N(s)}{N} - x(s) \right| > \epsilon \right) = 0$$

Similar result holds for hitting times

Comments on the Proof

Some ideas: R.W.R. Darling, J.R. Norris, *Differential equation approximations for Markov chains*, Probability Surveys, 5, pp. 37-79, 2008

Basic Ingredients

1. Use martingale decomposition and Doob's inequality to kill noise as $N \to \infty$
2. Apply Gronwall's lemma to control error propagation:
   - Gronwall's lemma:

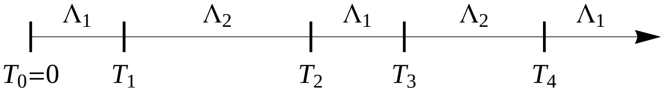
   $$f(t) \leq C + D \int_0^t f(s)ds \quad \Rightarrow \quad f(t) \leq Ce^{Dt}$$

   - M/M/∞ case: Gronwall can be directly applied (cf. Robert 2003)
   - More general case: As $N \to \infty$, the probability of being in a set on which condition (iii) holds goes to 1

Cyclically Varying Systems

Cyclically Varying Systems

- Back to the case: $\lambda, \mu X(t)^{\alpha}$
- A sequence of increasing time points $\{T_n, n \geq 0\}$
- Two sets of birth-death parameters $\Lambda_i = (\lambda_i, \mu_i)$, $i = 1, 2$
- At time points T_n, $X(t)$ changes behavior, alternating between Λ_1 and Λ_2



- Assume that the resulting process is a still a Markov process, possibly with a larger state space

Types of Cyclic Behavior

Hysteresis Control

$$T_n = \inf\{t > T_{n-1} : X(t) = \begin{cases} \ell_2 & n \text{ odd} \\ \ell_1 & n \text{ even} \end{cases} \}$$

Fixed Cycles

$$T_n - T_{n-1} = \begin{cases} \tau_1 & n \text{ odd} \\ \tau_2 & n \text{ even} \end{cases}$$

Random Environment

$$T_n - T_{n-1} \sim \begin{cases} \exp\left(\tau_1^{-1}\right) & n \text{ odd} \\ \exp\left(\tau_2^{-1}\right) & n \text{ even} \end{cases}$$

Related Literature

Hysteresis Control

Federgruen and Tijms 1980, Perry 1997, Bekker 2009...

Fixed Cycles

Harrison and Lemoine 1977, Lemoine 1989, Breuer 2004...

Random Environment

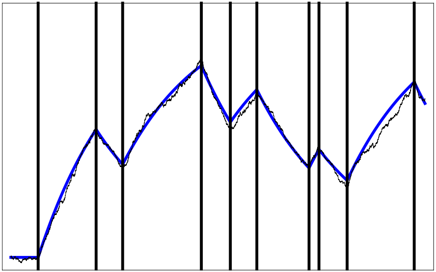
Yechiali and Naor 1971, Neuts 1977, Prabhu and Zhu 1989, Boxma and Kurkova 2000, Falin 2008, Fralix and Adan 2009...

In general, the queue level distribution is "tough". Things get "tougher" as one moves from $\alpha = 0$ to $\alpha = 1$ and then to arbitrary α.

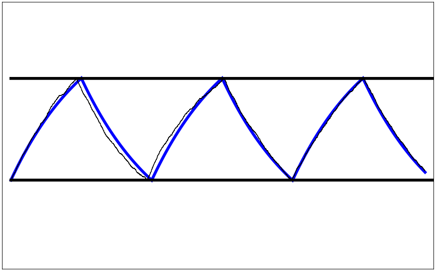
Approximating the Stationary Distribution

Basic Idea: Use the Scaling Limits

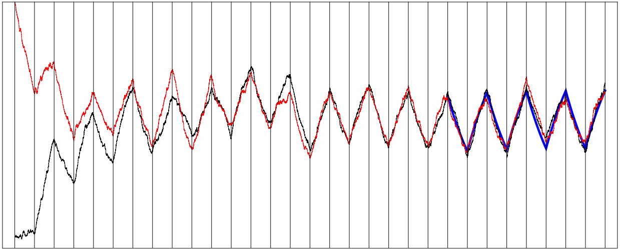
Random Environment



Hysteresis Control



Fixed Cycles



Basic Idea: Use the Scaling Limits

Hysteresis Control

Look at one deterministic cycle through $\ell_1 \to \ell_2 \to \ell_1$

Fixed Cycles

Look at one deterministic cycle of duration $\tau_1 + \tau_2$

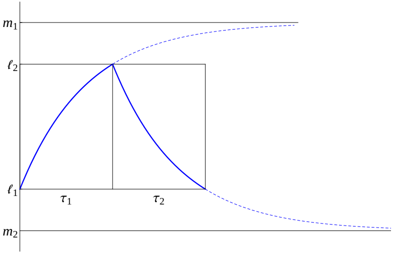
Random Environment

Look at a piece-wise deterministic Markov process (PDMP)

In all three cases we have a distribution function $F(\cdot)$ which is generated by the scaling limit

$F(\cdot)$ for Hysteresis Control and Fixed Cycles

$\dot{x}_i(t) = \lambda_i - \mu_i x(t)^\alpha$
$x_i(0) = \ell_i$
$\lim_{t\to\infty} x_i(t) = m_i$
$m_2 < \ell_1 < \ell_2 < m_1$
$\dot{x}_2(0) < 0 < \dot{x}_1(0)$
$\tau_i(y) = \inf\{t : x_i(t) = y\}$
$\tau_i = \tau_i(\ell_{\bar{i}})$



A CDF with support $[\ell_1, \ell_2]$, (assume $\alpha > 0$)

$$F(y) = \frac{1}{\tau_1 + \tau_2}\big(\tau_1(y) + (\tau_2 - \tau_2(y))\big)$$

- For Hysteresis control, ℓ_1, ℓ_2 given, τ_1, τ_2 easily calculated
- For Fixed Cycles τ_1, τ_2 given, unique ℓ_1, ℓ_2 obtained by solving:

$$x_1\Big|_{x_1(0)=\ell_1}^{(\tau_1)} = \ell_2, \qquad x_2\Big|_{x_2(0)=\ell_2}^{(\tau_2)} = \ell_1$$

$F(\cdot)$ for Random Environment

- PDMP: Environment Markov chain alternates between $1, 2$. Given a mode, trajectory is deterministic with "state-dependent" rates.
- O. Kella and W. Stadje, *Exact Results for a Fluid Model with State-Dependent Flow Rates*, Prob. in Eng. and Inform. Sci., 16, pp. 389-402, 2002.

Stationary Distribution

Solve for $p_1(\cdot), p_2(\cdot)$ on $y \in (m_2, m_1)$

$$
\begin{aligned}
(\lambda_1 - \mu_1 y^{\alpha}) p_1'(y) &= {\tau_2}^{-1} p_2(y) - {\tau_1}^{-1} p_1(y) \\
(\lambda_2 - \mu_2 y^{\alpha}) p_2'(y) &= {\tau_1}^{-1} p_1(y) - {\tau_2}^{-1} p_2(y) \\
p_1(m_2) = 0, &\qquad p_2(m_1) = \frac{\tau_2}{\tau_1 + \tau_2}
\end{aligned}
$$

$$F(y) = p_1(y) + p_2(y), \quad y \in (m_2, m_1)$$

Some Cases where $F(\cdot)$ is explicit

Hysteresis Control or Fixed Cycles where $\alpha = 1$

$$F(y) = \int_{-\infty}^{y} f(u)du, \qquad f(u) = \frac{\frac{(\mu_1-\mu_2)u+(\lambda_2-\lambda_1)}{(\mu_1 u-\lambda_1)(\mu_2 u-\lambda_2)}}{\log\left(\frac{\mu_1\ell_1-\lambda_1}{\mu_1\ell_2-\lambda_1}\right)^{\frac{1}{\mu_1}}\left(\frac{\mu_2\ell_2-\lambda_2}{\mu_2\ell_1-\lambda_2}\right)^{\frac{1}{\mu_2}}}\mathbf{1}_{\{\ell_1\le u\le \ell_2\}}$$

For fixed cycles set: $\ell_i = \frac{(e^{\tau_i\mu_i}-1)\frac{\lambda_i}{\mu_i}+(e^{\tau_{\bar{i}}\mu_{\bar{i}}}-1)\frac{\lambda_{\bar{i}}}{\mu_{\bar{i}}}e^{\tau_i\mu_i}}{e^{\tau_i\mu_i+\tau_{\bar{i}}\mu_{\bar{i}}}-1}$

Hysteresis Control or Fixed Cycles with $\alpha = 0$

Uniform distribution, sometimes with masses at the endpoints

Random Environment with $\alpha = 0$

Truncated exponential distribution with masses at m_1 and m_2

Random Environment with $\alpha = 1$

When $\mu_1 = \mu_2 = \tau_1 = \tau_2 = 1$, uniform on $[\lambda_2, \lambda_1]$. Otherwise, more complex explicit expression

Convergence of Stationary Distributions

Let $X_N(\cdot)$ be the scaled modulated process. Assume it is positive-recurrent. Then:

$$\lim_{N\to\infty} \sup_y \left| P\left(\frac{X_N(\infty)}{N} \leq y\right) - F(y) \right| = 0,$$

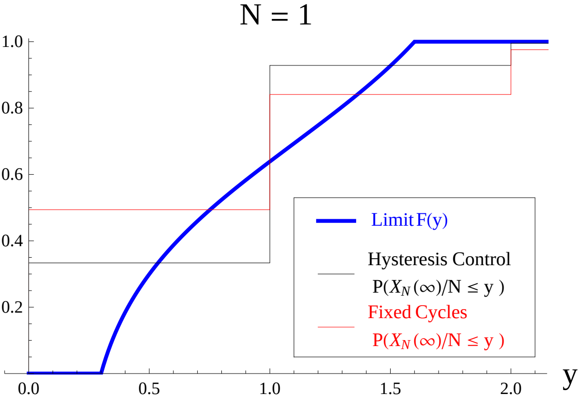
Note: For the N'th hysteresis control system use thresholds $\Big(\lceil N\ell_1 \rceil, \lfloor N\ell_2 \rfloor\Big)$

Approximating a "Real System"

– $\tilde{X}(\cdot)$ given with $\tilde{\Lambda}_i = \big(\tilde{\lambda}_i, \tilde{\mu}_i\big)$ and either $\tilde{\ell}_i$ or $\tilde{\tau}_i$
– Choose now N "smartly" (big enough) and set $\lambda_i = \tilde{\lambda}_i/N,\ \mu_i = \tilde{\mu}_i/N^{1-\alpha}$
– In case of hysteresis control set $\ell_i = \tilde{\ell}_i/N$
– $F(\cdot)$ now results from λ_i, μ_i and ℓ_i or τ_i or m_i as before
– The approximation: $$P(\tilde{X}(\infty) \leq y) \approx F\left(\frac{y}{N}\right)$$

Numerical Example: Hysteresis Control and Fixed Cycles

$$\alpha = 1 \quad \mu_1 = \mu_2 = 1 \quad \lambda_1 = 2 \quad \lambda_2 = 0.2$$
$$\ell_1 = 0.3 \quad \ell_2 = 1.6 \quad \tau_1 = 1.447 \quad \tau_2 = 2.639$$



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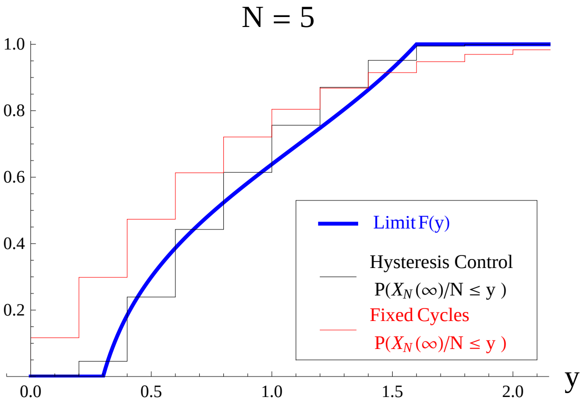

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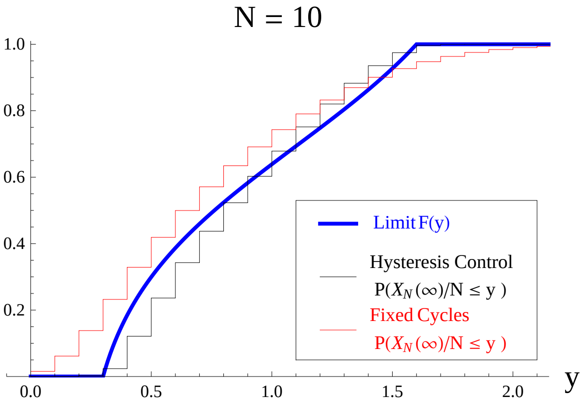

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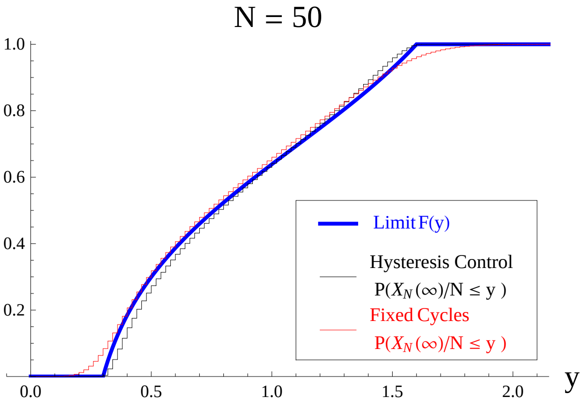

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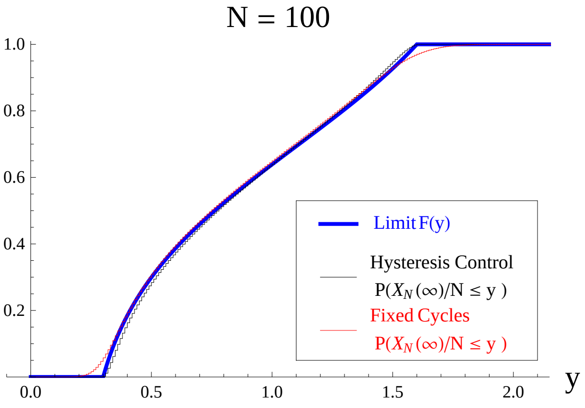

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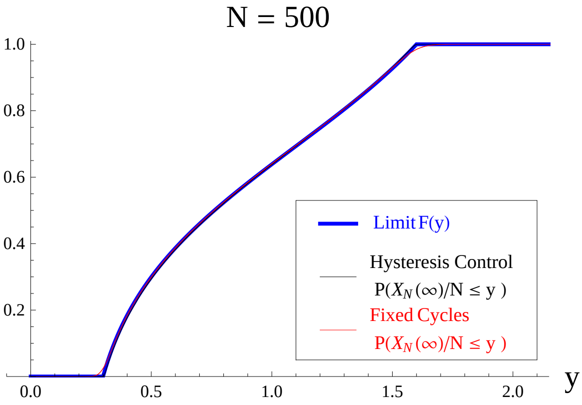
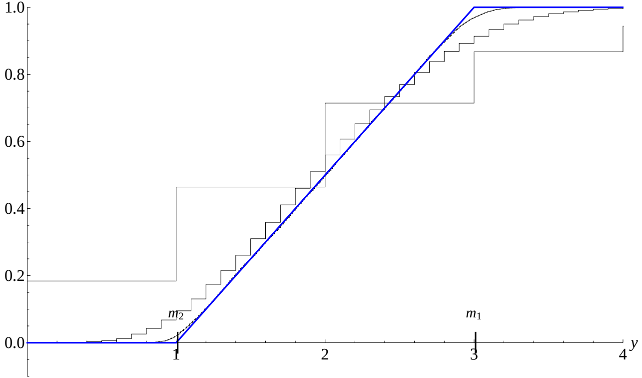


Numerical Example: Random Environment - Uniform

$$\alpha = 1$$

$$\mu_1 = \mu_2 = \tau_1 = \tau_2 = 1,\ \lambda_1 = 3,\ \lambda_2 = 1,$$

$$N = 1, 10, 100:$$

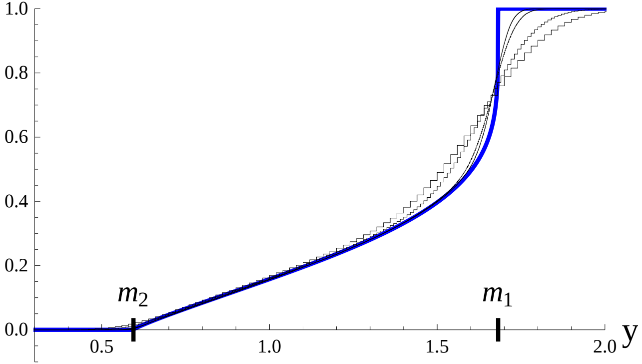


Numerical Example: Random Environment

$$\alpha = 4/3$$

$$\mu_1 = \mu_2 = 1,\ \lambda_1 = 2,\ \lambda_2 = 1/2,\ \tau_1 = 3,\ \tau_2 = 1$$

$$N = 50, 100, 500, 1000:$$



Conclusion

Contribution

- Extension of the well known M/M/1 and M/M/∞ scaling limits to more general birth-death processes
- A methodology for approximating stationary distributions of systems with cyclic behavior
- At the limit, hysteresis control and fixed cycles systems are indistinguishable
- New explicit asymptotic results for the M/M/∞ queue in a random environment

Further Work

- Rigorizing the stationary distribution convergence results
- Methods for choosing a "good" N for an approximation scheme
- Rates of convergence
- Approximately Optimizing "speed scaling systems" operating in cyclic environments
- Change point detection and prediction using the scaling limits

Questions