Stability of Queueing Networks with Infinite Supplies

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Based on some joint work with

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QANZIAM,
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The Study of Queueing Networks in Applied Probability and Operations Research

Congestion, delay and resource scarcity occurs
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- Customer service systems
- Complex manufacturing lines
- Telecommunication networks and computing systems
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Stochastic queueing network models capture the essential behaviour, allowing for quantitative performance evaluation, optimization and control

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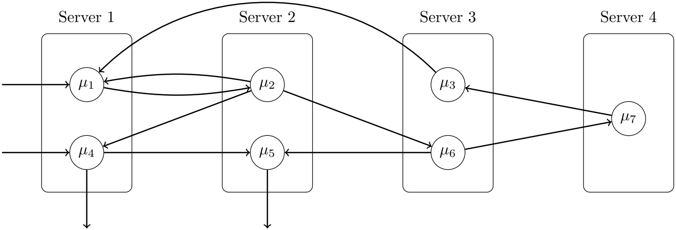
- $\rho = 1$ queue is **critically unstable**

Multi-Class Queueing Networks



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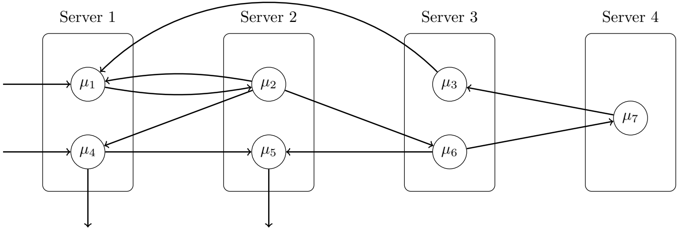

Feedback Control Policies

Policy: What operation should be served by each of the servers at every time instant based on the current state

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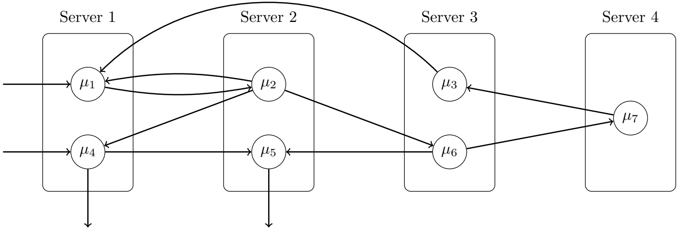

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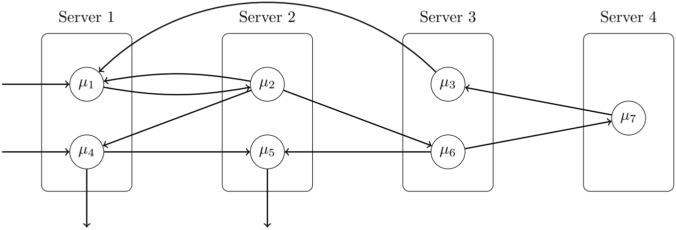

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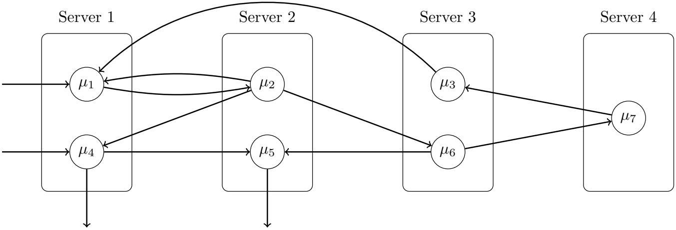

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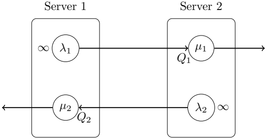
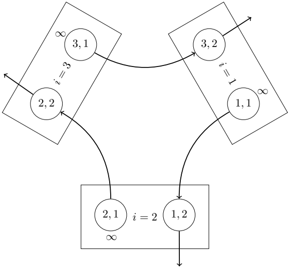
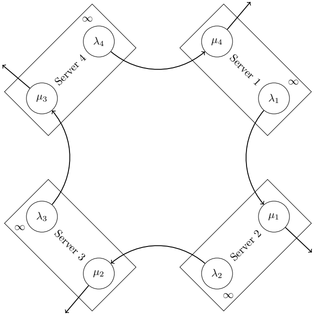
A realistic research goal:
understanding stability for given policies

Queueing Networks with Infinite Supplies

Push-Pull Rings with Pull-Priority Policy







Stochastic Model and Fluid Model

Stochastic Model $(Q(t), T(t))$

$$\begin{aligned} Q_i(t) &= Q_i(0) + S_{i,1}\big(T_{i,1}(t)\big) - S_{i,2}\big(T_{i,2}(t)\big) \\ t &= T_{i,1}(t) + T_{i-1,2}(t) \\ 0 &= \int_0^t Q_i(s)\,dT_{i+1,1}(s) \end{aligned}$$

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Associated Fluid Model $\big(\bar{Q}(t), \bar{T}(t)\big)$

$$
\begin{aligned}
\bar{Q}_i(t) &= \bar{Q}_i(0) + \lambda_i \bar{T}_{i,1}(t) - \mu_i \bar{T}_{i,2}(t) \\
t &= \bar{T}_{i,1}(t) + \bar{T}_{i-1,2}(t) \\
0 &= \int_0^t \bar{Q}_i(s) d\bar{T}_{i+1,1}(s)
\end{aligned}
$$

Fluid Stability Framework

Thm: (Dai '95), adapted to infinite supplies

Assume minor technical assumptions on the processing time distributions. If there exists a τ such that for all solutions of the fluid model and all $t \geq \tau$, $\sum \bar{Q}_i(t) = 0$ then the (stochastic) network is stable.

- All solutions of the fluid model are Lipschitz, thus have derivatives a.e.
- **Regular time points**: Time points at which derivatives exists

Lemma

If we have a Lyapounov function: $V : \mathbb{R}^M \rightarrow \mathbb{R}$ such that for all regular time points of all solutions of the fluid model, $\frac{d}{dt} V\left(\bar{Q}(t)\right) < -\epsilon$ for some $\epsilon > 0$, then the fluid model is stable.

Stability Result: M odd, $\gamma_i := \frac{\lambda_i}{\mu_i} > 1$

Theorem (Erjen Lefeber, Gideon Weiss, Y.N.)

The push-pull ring with M **odd**, $\gamma_i > 1$ for all i, operating under a pull-priority policy is stable if $\Delta < 0$, where

$$\Delta = \sum_{i=1}^{M} c_i \Big(\frac{M-1}{2} (\gamma_i - 1) - 1 \Big),$$

with,

$$c_i = ((\cdots(((\gamma_{i-1} - 1)\gamma_{i-2} + 1)\gamma_{i-3} - 1)\gamma_{i-4} \cdots\cdots\cdots)\gamma_{i+2} - 1)\gamma_{i+1} + 1.$$

Note: If $\gamma_i = \gamma$ for all i then the stability condition reduces to:

$$\gamma < 1 + \frac{1}{\frac{M-1}{2}}$$

Proof Overview

1. Use $V(x) = \sum_{i=1}^{M} c_i\, x_i$ as Lyapounov function for the fluid model with coefficients, c_i, designed based on the intuition that "typical" fluid trajectories eventually cycle on states of the form (e.g. $M = 5$):

$$(+,0,+,0,+),\ \ (+,+,0,+,0),\ \ (0,+,+,0,+),\ \ (+,0,+,+,0),\ \ (0,+,0,+,+).$$

The c_i are such that $\dot{V}(t)$ is constant during such cycles:

$$\begin{bmatrix} -1 & 0 & \gamma_3 - 1 & 0 & \gamma_5 - 1 \\ \gamma_1 - 1 & -1 & 0 & \gamma_4 - 1 & 0 \\ 0 & \gamma_2 - 1 & -1 & 0 & \gamma_5 - 1 \\ \gamma_1 - 1 & 0 & \gamma_3 - 1 & -1 & 0 \\ 0 & \gamma_2 - 1 & 0 & \gamma_4 - 1 & -1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \\ c_5 \end{bmatrix} = \begin{bmatrix} \Delta \\ \Delta \\ \Delta \\ \Delta \\ \Delta \end{bmatrix}$$

2. Characterize the regular time points and show that on these time points the Lyapounov function has negative drift
3. Now apply Jim Dai's stability framework

The End