A Problem

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Assume:

- $\mathcal{Q}$ is a regenerative process with epochs $\{\tau_1, \tau_2, \ldots\}$.
- $\{D(t), t \geq 0\}$ and $\{C(t), t \geq 0\}$ are non-decreasing processes with RCLL sample paths. Defined on the same probability space as $\mathcal{Q}$.
- $C(t)$ is constant over $t \in [\tau_n, \tau_{n+1})$.
- $C(\tau(t)) = D(\tau(t))$ w.p. 1, where $\tau(t) = \sup\{\tau_1, \tau_2, \ldots | \tau_n \leq t\}$.
- $\{D(\tau_{n+1}) - D(\tau_n), n \geq 1\}$ i.i.d.
- $\mathbb{E}\,[(D(\tau_2) - D(\tau_1))^2],\ \mathbb{E}\,[(\tau_2 - \tau_1)^2] < \infty$.
- There exist constants $\overline{V}_C, \overline{B}_C, \overline{V}_D, \overline{B}_D$:

$$\mathrm{Var}(D(t)) = \overline{V}_D t + \overline{B}_D + o(1) \quad \text{and} \quad \mathrm{Var}(C(t)) = \overline{V}_C t + \overline{B}_C + o(1).$$

Then,

(i) $\overline{V}_D = \overline{V}_C$.

(ii) $\overline{B}_D = \overline{B}_C + \overline{\Delta}$.
Where $\overline{\Delta}$ is an expression of the moments of $\tau_2 - \tau_1$ and $D(\tau_2) - D(\tau_1)$.

The proof of (i) is simple. Don't know about (ii) - possibly it can not be done (to find such a $\overline{\Delta}$). Proof? Counter-example?