QPA Reading Seminar

Stochastic Storage Processes by N.U. Prabhu

Chapter 6: A Fluid Model for Data Communication

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The Model

$\{J(t), t \geq 0\}$ is a non-explosive irreducible CTMC on state space $\mathcal{E}$.

Two non-negative rate functions: $a, d : \mathcal{E} \to \mathbb{R}_+$.

$a(j)$ - inflow. $d(j)$ - outflow capacity.

$$r(z,j) = \begin{cases} d(j) & z > 0 \\ \min(a(j), d(j)) & z \leq 0 \end{cases}$$

$$Z(t) = Z(0) + \int_0^t a(J(s))ds - \int_0^t r(Z(s), J(s))ds.$$

It is a "water tank" with inflow rate $a(J(s))$ and outflow pumping capability of rate $d(J(s))$.

Example Applications

Example 1:

$$a(j) = j, \; d(j) = c.$$

Example 2:

$$a(j) = c_0, \; d(j) = c_1 j.$$

Example 3:

$$a(j) = c_0 1_{\{j<0\}}, \; d(j) = c_1.$$

Preview of One of the Main Results (of Practical Interest)

Theorem 9 (ii): If $\sum_{j \in \mathcal{E}} \pi_j (a(j) - d(j)) < 0$ then,

$$P(Z_\infty = 0 | J_\infty = k) = \begin{cases} 0 & a(k) > d(k) \\ E[e^{\frac{-\nu_{kk}}{a(k)-d(k)} Z^*_\infty} | J^*_\infty = k] & a(k) \leq d(k) \end{cases} .$$

Where (Z^*, J^*) is the embedded Markov chain and $E[e^{i\omega Z^*_\infty} | J^*_\infty]$ is given in Theorem 7 (iii) by the k'th element of the row vector:

$$\pi^* [I - \chi(1, 0, 0)][I - \chi(1, 0, \omega)]^{-1}.$$

Here the matrix χ of "ladder height transforms" is characterized in Theorem 5 (not fully computable?).

A Reflection of Net Input

Net arrivals and net outflow capacity (demand):

$$A(t) = \int_0^t a(J(s))ds, \qquad D(t) = \int_0^t d(J(s))ds.$$

Net input (our unreflected "random walk"):

$$X(t) = \int_0^t x(J(s))ds, \text{ with } x(j) = a(j) - d(j).$$

Rewriting $Z(t)$ (note that $r(z, j) = d(j) + \min(x(j), 0)1_{\{z \leq 0\}}$):

$$Z(t) = Z(0) + X(t) + I(t), \text{ with } I(t) = -\int_0^t \min(0, x(J(s)))1_{\{Z(s) \leq 0\}} ds.$$

I(t) is waisted outflow pumping capacity (unsatisfied demand).

Reflection Mapping is Unique

Theorem 1: The integral equation

$$Z(t) = Z(0) + X(t) - \int_0^t \min(0, x(J(s)))1_{\{Z(s)\le 0\}} ds$$

has a unique non-negative solution

$$Z(t) = Z(0) + X(t) + I(t)$$

where

$$I(t) = -\min(0, Z(0) + \inf_{0\le s\le t} X(s)).$$

Embedded Time Points

Denote also,

$$m(t) = \inf_{0 \le s \le t} X(s), \qquad M(t) = \sup_{0 \le s \le t} X(s).$$

Denote $\{T_n, n = 0, 1, \dots\}$ - the transitions epochs of J,

$$S_n = X(T_n), \quad Z_n = Z(T_n), \quad I_n = I(T_n), \quad m_n = m(T_n), \quad M_n = M(T_n).$$

For $t \in [T_n, T_{n+1})$:

$$\begin{aligned} Z(t) &= Z_n + x(J(T_n))(t - T_n) - \min(0, x(J(T_n))) \int_{T_n}^{t} 1_{\{Z(s) \le s\}} ds, \\ I(t) &= I_n - \min(0, x(J(T_n))) \int_{T_n}^{t} 1_{\{Z(s) \le 0\}} ds. \end{aligned}$$

Discrete Time Reflection Mapping Does the Job

Theorem 2: For $t \in [T_n, T_{n+1})$:

$Z(t) = \max(0, Z_n + x(J_n)(t - T_n)), \qquad I(t) = I_n - \min(0, Z_n + x(J_n)(t - T_n)),$

where,

$$Z_n = Z_0 + S_n + I_n, \qquad I_n = -\min(0, Z_0 + m_n).$$

The Net Input Process is a MAP

$X(t)$ was called the net input. Now call $(X, J) = \{X(t), J(t)\}$ the net input process.

Since for $t < t'$,

$$X(t') = X(t) + \int_t^{t'} x(J(s))ds,$$

we have that given $\{X(s), J(s), 0 \leq s \leq t\}$, the distribution of $\{X(t'), J(t')\}$ depends only on $\{X(t), J(t)\}$.

Also, given $\{X(t), J(t)\}$, the conditional distribution of $\{X(t') - X(t), J(t')\}$ depends only on $J(t)$.

Thus, (X, J) is a MAP (Markov Additive Process) on the state space $\mathbb{R} \times \mathcal{E}$.

The Mean Net Input Rate (Mean Drift)

Partition $\mathcal{E}$:

$$\mathcal{E}_0 = \{j | x(j) > 0\}, \qquad \mathcal{E}_1 = \{j | x(j) \leq 0\}.$$

Let (ν_{jk}) be the generator matrix of J and assume a (unique) stationary distribution (π_j).

The mean net input rate is,

$$\overline{x} = \sum_{j \in \mathcal{E}} \pi_j x(j).$$

(assume the sum exists - maybe infinite).

Some Non-Surprising Properties of the Net Input Process

Lemma 1: w.p. 1:

$$\lim_{t\to\infty} \frac{X(t)}{t} \to \overline{x}.$$

Theorem 3: w.p. 1:

	$\lim_{t\to\infty} X(t)$	m	M
$\overline{x} < 0$	$= -\infty$	$= -\infty$	$< \infty$
$\overline{x} = 0$	$\liminf = -\infty, \limsup = \infty$	$= -\infty$	$= \infty$
$\overline{x} > 0$	$= \infty$	$-\infty <$	$= \infty$

Introducing the MRW $\{T_n, S_n, J_n\}$

The embedded Markov chain $J^* = \{J_n, n \geq 0\}$ has transition probabilities (P_{jk}) and stationary distribution (π_j^*):

$$P_{jk} = \frac{\nu_{jk}}{-\nu_{jj}} 1_{\{j \neq k\}}, \quad \pi_j^* = \frac{-\nu_{jj}}{\sum_{k \in \mathcal{E}} (-\nu_{kk}) \pi_k} \pi_j.$$

Denote,

$$\mu^* = \sum_{j \in \mathcal{E}} \pi_j^* E[X_1 | J_0 = j] = \frac{\overline{x}}{\sum_{k \in \mathcal{E}} \pi_k (-\nu_{kk})}.$$

The Time Reversal of the Net Input Process

We use the time reversal of the MRW to define the time reversal of the net input process: $(\hat{X}, \hat{J})$.

First define $(\hat{T}_n, \hat{S}_n, \hat{J}_n)$ as the time-reversed MRW of (T_n, S_n, J_n):

Take $(\hat{J}_n)$ as a DTMC with transition rates:

$$\hat{P}_{jk} = \frac{\pi_k^*}{\pi_j^*} P_{kj},$$

and with transition times $(\hat{T}_n)$. Now, $\hat{S}_n = \hat{X}(\hat{T}_n)$, where $\hat{X}$ is a net input modulated by $(\hat{J}_n)$.

To get $(\hat{X}, \hat{J})$, set $(\hat{\nu}_{jj}) = (\nu_{jj})$.

Joint Transform for First Step

$$\begin{aligned}\phi_{jk}(\theta,\omega) &= E[e^{-\theta T_1 + i\omega X_1} 1_{\{J_1=k\}} | J_0 = j] \\ &= E[e^{-(\theta - i\omega x(j))T_1} 1_{\{J_1=k\}} | J_0 = j] \\ &= \frac{\nu_{jk}}{-\nu_{jj} + \theta - i\omega x(j)} I_{\{j\neq k\}}. \\ \hat{\phi}_{jk}(\theta,\omega) &= E[e^{-\theta \hat{T}_1 + i\omega \hat{X}_1} 1_{\{\hat{J}_1=k\}} | \hat{J}_0 = j] \\ &= \frac{\pi_k^*}{\pi_j^*}\phi_{kj}(\theta,\omega).\end{aligned}$$

Define the matrices $\Phi = (\phi_{jk}(\theta, w))$ and $\hat{\Phi} = (\hat{\phi}_{jk}(\theta, w))$. Then $\Phi = \alpha P$ and $\hat{\Phi} = \hat{P}\alpha$, with α being a diagonal matrix α with j-th diagonal entry

$$\alpha_j = \frac{-\nu_{jj}}{-\nu_{jj} + \theta - i\omega x(j)}.$$

Ladder Epochs of the MRW

Define the epochs of (T_n, S_n, J_n) and $(\hat{T}_n, \hat{S}_n, \hat{J}_n)$:

$$\begin{aligned}\bar{N} &= \min(n : S_n < 0) \quad &&\text{– (Descending ladder epoch)}\\ N &= \min(n : \hat{S}_n > 0) \quad &&\text{– (Ascending ladder epoch of reversal)}\end{aligned}$$

The corresponding transform matrices $\bar{\chi} = (\bar{\chi}_{jk}(z, \theta, w))$ and $\chi = (\chi_{jk}(z, \theta, w))$ with,

$$\begin{aligned}\bar{\chi}_{jk}(z, \theta, w) &= E\big[z^{\bar{N}} e^{-\theta T_{\bar{N}} + iwS_{\bar{N}}} \mathbf{1}_{\{J_{\bar{N}}=k\}} | J_0 = j\big],\\ \chi_{jk}(z, \theta, w) &= \frac{\pi_k^*}{\pi_j^*} E\big[z^N e^{-\theta \hat{T}_N + iw\hat{S}_N} \mathbf{1}_{\{\hat{J}_N=j\}} | \hat{J}_0 = k\big].\end{aligned}$$

Ladder Epochs of the Net Input Process

Define the epochs of (X, J) and $(\hat{X}, \hat{J})$:

$$\begin{aligned} \bar{T} &= \inf(t > 0 : X(t) \leq 0) \quad &\text{– (Descending ladder epoch)} \\ T &= \inf(t > 0 : \hat{X}(t) \geq 0) \quad &\text{– (Ascending ladder epoch of the reversal)} \end{aligned}$$

The corresponding transform matrices are $\xi = (\xi_{jk}(z, \theta))$ and $\eta = (\eta_{jk}(z, \theta))$ with,

$$\begin{aligned} \xi_{jk}(z, \theta) &= E\big[z^{\bar{N}} e^{-\theta \bar{T}} \mathbf{1}_{\{J(\bar{T})=k\}} | J(0) = j\big], \\ \eta_{jk}(z, \theta) &= \frac{\pi_k^*}{\pi_j^*} E\big[z^{N} e^{-\theta T} \mathbf{1}_{\{\hat{J}(T)=j\}} | \hat{J}(0) = k\big]. \end{aligned}$$

Relating the Transforms of the MRW and Net Input Process

Theorem 4:

$$\bar{\chi} = \xi \alpha P, \quad \chi = \alpha \eta.$$

This theorem relates the ladder epoch and height transforms of the MRW (discrete time) to the transforms of the the Net Input Process (continuous time).

Block Matrix Notation

Rearrange the matrix P as follows:

$$P = \left(\begin{array}{c|c} P_{00} & P_{01} \\ \hline P_{10} & P_{11} \end{array} \right),$$

where P_{ij} contains transition probabilities between states in $\mathcal{E}_i$ and states in $\mathcal{E}_j$, $i, j = 0, 1$.

Do the same rearrangement for the matrices α, ξ and η.

Reminder:

$$\mathcal{E}_0 = \{j | x(j) > 0\}, \qquad \mathcal{E}_1 = \{j | x(j) \leq 0\}.$$

Key Theorem: Applying Wiener-Hopf Factorization of the MRW

Theorem 5:

For $0 < z < 1$, $\theta > 0$ and ω real,

$$\chi = \left(\begin{array}{c|c} \alpha_{00}\eta_{00} & \alpha_{00}\eta_{01} \\ \hline 0 & 0 \end{array} \right), \quad \bar{\chi} = \left(\begin{array}{c|c} \xi_{01}\alpha_{11}P_{10} & \xi_{01}\alpha_{11}P_{11} \\ \hline z\alpha_{11}P_{10} & z\alpha_{11}P_{11} \end{array} \right),$$

and

$$\begin{aligned} I - \bar{\chi}_{00} &= (I - \chi_{00})^{-1}(I - z\alpha_{00}P_{00} - z\chi_{01}\alpha_{11}P_{01}), \\ \bar{\chi}_{01} &= (I - \chi_{00})^{-1}(z\alpha_{00}P_{01} + z\chi_{01}\alpha_{11}P_{11} - \chi_{01}). \end{aligned}$$

where the inverse exists in the specified domain.

Proof uses Lemma 2:

For $0 < z < 1$ and $\theta > 0$:

$$\xi = \left(\begin{array}{c|c} 0 & \xi_{01} \\ \hline 0 & zI \end{array}\right), \quad \eta = \left(\begin{array}{c|c} \eta_{00} & \eta_{01} \\ \hline 0 & 0 \end{array}\right).$$

The Busy (non-empty) Period

Assume $Z(0) = 0$, w.p. 1. Then,

$$\overline{T} = \inf\{t > 0 | X(t) \leq 0\} = \inf\{t > 0 | Z(t) = 0\},$$

is defined as the busy period (non empty period) of the storage process $Z(t)$.

$$E[e^{-\theta \overline{T}} 1_{\{J(\overline{T})=k\}} | J(0) = j] = \xi_{jk}(1, \theta).$$

Theorem 6:

(i) Given $J(0) \in \mathcal{E}_1$, then $\overline{T} = 0$, w.p. 1.

(ii) If $\overline{x} < 0$ then $\overline{T} < \infty$ w.p. 1.

Using the MRW to Analyze $\{Z_n, I_n, J_n\}$

Theorem 7: Assume $Z_0 = 0$:

(i)

$$\left(\sum_{n=0}^{\infty} z^n E[e^{-\theta T_n + i\omega_1 Z_n + i\omega_2 I_n} 1_{\{J_n = k\}} | J_0 = j]\right)^{-1}$$
$$= [I - \chi(z, \theta, \omega_1)][I - \bar{\chi}(z, \theta, -\omega_2)].$$

(ii) If $\mu^* > 0$ then as $n \to \infty$, (I_n, J_n) converges in distribution to (I^*_∞, J^*_∞) with,

$$\left(E[e^{iwI^*_\infty} 1_{\{J^*_\infty = k\}} | J^*_0 = j]\right) = [I - \bar{\chi}(1, 0, -\omega)]^{-1}[I - \bar{\chi}(1, 0, 0)]e'\pi^*.$$

(iii) If $\mu^* < 0$ then as $n \to \infty$, (Z_n, J_n) converges in distribution to (Z^*_∞, J^*_∞), where, $E[e^{i\omega Z^*_\infty} 1_{\{J^*_\infty = k\}}]$, is the k'th element of,

$$\pi^*[I - \chi(1, 0, 0)][I - \chi(1, 0, \omega)]^{-1}.$$

Moving Back to $Z(t)$ and $I(t)$ (Continuous Time)

First Some Non-Surprising Results

Theorem 8: (i) w.p. 1:

$$\lim_{t\to\infty} I(t) = -\min(0, Z(0) + \inf_{0\le s<\infty} X(s)) < \infty,$$

if and only if $\overline{x} > 0$.

(ii) w.p. 1:

$$\begin{aligned}
\lim_{t\to\infty} \frac{Z(t)}{t} &= 0 \qquad \text{if } \overline{x} < 0 \\
\limsup_{t\to\infty} Z(t) &= \infty \qquad \text{if } \overline{x} = 0 \\
\lim_{t\to\infty} Z(t) &= \infty \qquad \text{if } \overline{x} > 0
\end{aligned}$$

First Main Result
(of Practical Importance)

Denote (Z_∞, J_∞) the limiting random variable of $(Z(t), J(t))$ when it exists.

Theorem 9: If $\overline{x} < 0$ then (Z_∞, J_∞) exists and,

(i)

$$P(Z_\infty \leq z, J_\infty = k) =$$
$$\pi_k \int_{0^-}^{\infty} P(Z^*_\infty \in du | J^*_\infty = k) P(X_1 \leq z - u | J(0) = k),$$

(ii)

$$P(Z_\infty = 0 | J_\infty = k) = \begin{cases} 0 & a(k) > d(k) \\ E[e^{\frac{-\nu_{kk}}{a(k)-d(k)} Z^*_\infty} | J^*_\infty = k] & a(k) \leq d(k) \end{cases}.$$

Corollary to the Main Result (of Practical Importance)

Define $I_k(t)$ as the unsatisfied demand when J is in state k during $(0, t]$:

$$I_k(t) = -\int_0^t \min(0, x(J(s)))1_{\{Z(s)=0, J(s)=k\}} ds.$$

Theorem 10: Assume $\overline{x} < 0$ and $k \in \mathcal{E}_1$:

$$\text{(i)} \quad \lim_{t\to\infty} \frac{I_k(t)}{t} = -x(k)\pi_k E[e^{\frac{-\nu_{kk}}{x(k)} Z^*_\infty} | J^*_\infty = k].$$

$$\text{(ii)} \quad \lim_{t\to\infty} \frac{I_k(t)}{I(t)} = \frac{x(k)\pi_k E[e^{\frac{-\nu_{kk}}{x(k)} Z^*_\infty} | J^*_\infty = k]}{\sum_{j\in\mathcal{E}_1} x(j)\pi_j E[e^{\frac{-\nu_{jj}}{x(j)} Z^*_\infty} | J^*_\infty = j]}.$$

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