Duality in Conic Programming

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This brief article introduces the conic programming problem in standard form and describes some results and challenges associated with duality.

1 The Conic Programming Problem and its Dual

A standard conic programming problem, \([1]\), takes the form

\[
\begin{align*}
\min_{x \in \mathbb{R}^n} \langle c, x \rangle \\
\text{s.t.} \quad Ax &= b \\
\quad x &\in K,
\end{align*}
\]

where \(c \in \mathbb{R}^n\), \(b \in \mathbb{R}^m\), \(A \in \mathbb{R}^{n \times m}\), \(\langle \cdot, \cdot \rangle\) denotes the standard Euclidean inner product and \(K \subset \mathbb{R}^n\) is a convex cone. That is, \(K\) is a convex set with the property that for all \(x \in K\), \(\lambda x \in K\) for all \(\lambda > 0\).

The corresponding dual problem is

\[
\begin{align*}
\max_{y \in \mathbb{R}^m} \langle b, y \rangle \\
\text{s.t.} \quad A^*y + s &= c \\
\quad s &\in K^*,
\end{align*}
\]

where \(A^*\) denotes the transpose of \(A\) and \(K^*\) is the dual cone \([3]\), defined by

\(K^* = \{s \in \mathbb{R}^n : \langle s, x \rangle \geq 0, \forall x \in K\}\).

2 Deriving the Dual Problem

We will treat the problem as a constrained optimization problem in order to derive the dual. Let \(y\) be the Lagrange multiplier associated with the constraint \(Ax = b\). Then the Lagrange dual problem of (P) is

\[
L(y) = \inf_{x \in K} \langle c, x \rangle + \langle y, b - Ax \rangle
= \langle y, b \rangle + \inf_{x \in K} \left[ \langle c, x \rangle - \langle y, Ax \rangle \right]
= \langle y, b \rangle + \inf_{x \in K} \langle c - A^*y, x \rangle
\]

\([4]\).
If $c - A^* y \in K^*$, then $\langle c - A^* y, x \rangle \geq 0$, so $\inf_{x \in K} \langle c - A^* y, x \rangle = 0$. If $c - A^* y \notin K^*$, then $\inf_{x \in K} \langle c - A^* y, x \rangle = -\infty$. Thus, $L(y)$ is given by

$$L(y) = \begin{cases} 
(y, b), & c - A^* y \in K^*, \\
-\infty, & c - A^* y \notin K^*.
\end{cases}$$

Let $s = c - A^* y \in K^*$. Then the Lagrangian dual problem is to maximise $L(y)$, which can be written as \((D)\).

### 3 Weak Duality

Let $x$ be a feasible solution to \(P\) and $y$ a feasible solution to \(D\). Then we say weak duality holds if $\langle c, x \rangle \geq \langle b, y \rangle$. So a lower bound on the optimal solution of the primal problem can be obtained by solving the dual problem. We have the following result:

**Theorem 1.** Let $x$ be a feasible solution to \(P\) and $y$ be a feasible solution to \(D\). Then weak duality holds.

**Proof.**

$$\langle c, x \rangle = \langle A^* y + s, x \rangle = \langle A^* y, x \rangle + \langle s, x \rangle = \langle y, A x \rangle + \langle s, x \rangle = \langle y, b \rangle + \langle s, x \rangle \geq \langle b, y \rangle,$$

where $\langle s, x \rangle \geq 0$ follows since $s \in K^*$.

### 4 Strong Duality

If it happens that $\langle c, x \rangle = \langle b, y \rangle$ where $x$ is the optimal solution to \((P)\) and $y$ is the optimal solution to \((D)\), then we say we have strong duality. Unlike linear programming, strong duality does not always hold in conic programming [5]. To see this, consider the following counterexample. Let $K$ be the cone

$$K = \{ x \in \mathbb{R}^4 : x_1^2 + x_2^2 \leq x_3^2, x_3, x_4 \geq 0 \},$$

and note that $K = K^*$, so $K$ is a self dual cone. Let the primal problem be

$$\begin{align*}
\min & \quad -x_1 \\
\text{s.t.} & \quad x_1 + x_4 = 1 \\
& \quad x_2 + x_3 = 0 \\
& \quad x \in K.
\end{align*}$$

Observe that for a feasible point $(x_1, x_2, x_3, x_4) \in K$, then

$$x_1^2 \leq x_3^2 - x_2^2 = (x_3 - x_2)(x_3 + x_2) = 0,$$

as $x_2 + x_3 = 0$ in the primal problem, so $x_1 = 0$. Therefore $(0,0,0,1)$ is an optimal solution to the primal problem with objective value 0.
Using the definition from above, the dual problem is

\[
\begin{align*}
\max & \quad y_1 \\
\text{s.t.} & \quad y_1 + s_1 = -1 \\
& \quad y_2 + s_2 = 0 \\
& \quad y_2 + s_3 = 0 \\
& \quad y_1 + s_4 = 0 \\
(s_1, s_2, s_3, s_4) & \in K^*. 
\end{align*}
\]

Representing \( y \) in terms of \( s \) and using the fact that \( K \) is self dual, the dual problem is equivalent to

\[
\begin{align*}
\max & \quad y_1 \\
\text{s.t.} & \quad -(1 + y_1, y_2, y_2, y_1) \in K. 
\end{align*}
\]

This implies that

\[
(1 + y_1)^2 + y_2^2 \leq y_2^2 \quad \text{and} \quad y_1, y_2 \leq 0.
\]

The first inequality implies \((1 + y_1)^2 \leq 0\), which gives \(y_1 = -1\). So \((-1, 0)\) is an optimal solution to the dual problem with an objective value \(-1\). Therefore the optimal objective of the primal problem does not equal the optimal objective of the dual. Hence strong duality does not always hold.

### 5 Slater Constraint Qualification

The following theorem provides a sufficient condition for strong duality in conic programming problems.

**Theorem 2.** If there exists a point \( x \) in the relative interior of \( K \) such that \( Ax = b \), then if the primal problem has an optimal solution, then the dual problem has an optimal solution, and the optimal values are equal, so strong duality holds.

See [2] for a proof of this theorem.

Note that for the counterexample to strong duality above, there are no points \( x \) in the relative interior of \( K \) such that \( Ax = b \), so we cannot apply the Slater constraint qualification.

### References


