Duality in Conic Programming

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This brief article introduces the conic programming problem in standard form and describes some results and challenges associated with duality.

1 The Conic Programming Problem and its Dual

A standard conic programming problem, [1], takes the form

$$\begin{aligned} \min_{x\in\mathbb{R}^n} \quad & \langle c, x\rangle \\ \text{s.t.} \quad & Ax = b \qquad (P) \\ & x \in K, \end{aligned}$$

where $c \in \mathbb{R}^n$, $b \in \mathbb{R}^m$, $A \in \mathbb{R}^{n\times m}$, $\langle\cdot,\cdot\rangle$ denotes the standard Euclidean inner product and $K \subset \mathbb{R}^n$ is a convex cone. That is, K is a convex set with the property that for all $x \in K$, $\lambda x \in K$ for all $\lambda > 0$.

The corresponding dual problem is

$$\begin{aligned} \max_{y\in\mathbb{R}^m} \quad & \langle b, y\rangle \\ \text{s.t.} \quad & A^* y + s = c \qquad (D) \\ & s \in K^*, \end{aligned}$$

where A^* denotes the transpose of A and K^* is the dual cone [3], defined by

$$K^* = \{s \in \mathbb{R}^n \ : \ \langle s, x\rangle \geq 0, \ \forall x \in K\}.$$

2 Deriving the Dual Problem

We will treat the problem as a constrained optimization problem in order to derive the dual. Let y be the Lagrange multiplier associated with the constraint $Ax = b$. Then the Lagrange dual problem of (P) is

$$\begin{aligned} L(y) &= \inf_{x\in K} \langle c, x\rangle + \langle y, b - Ax\rangle \\ &= \langle y, b\rangle + \inf_{x\in K} \left[\langle c, x\rangle - \langle y, Ax\rangle\right] \\ &= \langle y, b\rangle + \inf_{x\in K} \langle c - A^* y, x\rangle \qquad [4]. \end{aligned}$$

If $c - A^*y \in K^*$, then $\langle c - A^*y, x\rangle \geq 0$, so $\inf_{x\in K} \langle c - A^*y, x\rangle = 0$. If $c - A^*y \notin K^*$, then $\inf_{x\in K} \langle c - A^*y, x\rangle = -\infty$. Thus, $L(y)$ is given by

$$L(y) = \begin{cases} \langle y, b\rangle, & c - A^*y \in K^*, \\ -\infty, & c - A^*y \notin K^*. \end{cases}$$

Let $s = c - A^*y \in K^*$. Then the Lagrangian dual problem is to maximise $L(y)$, which can be written as (D).

3 Weak Duality

Let x be a feasible solution to P and y a feasible solution to D. Then we say weak duality holds if $\langle c, x\rangle \geq \langle b, y\rangle$. So a lower bound on the optimal solution of the primal problem can be obtained by solving the dual problem. We have the following result:

Theorem 1. *Let x be a feasible solution to P and y be a feasible solution to D. Then weak duality holds.*

Proof.

$$\langle c, x\rangle = \langle A^*y + s, x\rangle = \langle A^*y, x\rangle + \langle s, x\rangle = \langle y, Ax\rangle + \langle s, x\rangle = \langle y, b\rangle + \langle s, x\rangle \geq \langle b, y\rangle,$$

where $\langle s, x\rangle \geq 0$ follows since $s \in K^*$. □

4 Strong Duality

If it happens that $\langle c, x\rangle = \langle b, y\rangle$ where x is the optimal solution to (P) and y is the optimal solution to (D), then we say we have strong duality. Unlike linear programming, strong duality does not always hold in conic programming [5]. To see this, consider the following counterexample. Let K be the cone

$$K = \{x \in \mathbb{R}^4 \ : \ x_1^2 + x_2^2 \leq x_3^2, \ x_3, x_4 \geq 0\},$$

and note that $K = K^*$, so K is a self dual cone. Let the primal problem be

$$\begin{aligned} \min \quad & -x_1 \\ \text{s.t.} \quad & x_1 + x_4 = 1 \\ & x_2 + x_3 = 0 \\ & x \in K. \end{aligned}$$

Observe that for a feasible point $(x_1, x_2, x_3, x_4) \in K$, then

$$\begin{aligned} x_1^2 &\leq x_3^2 - x_2^2 \\ &= (x_3 - x_2)(x_3 + x_2) \\ &= 0, \end{aligned}$$

as $x_2 + x_3 = 0$ in the primal problem, so $x_1 = 0$. Therefore $(0, 0, 0, 1)$ is an optimal solution to the primal problem with objective value 0.

Using the definition from above, the dual problem is

$$\begin{aligned} \max \quad & y_1 \\ & y_1 + s_1 = -1 \\ & y_2 + s_2 = 0 \\ & y_2 + s_3 = 0 \\ & y_1 + s_4 = 0 \\ & (s_1, s_2, s_3, s_4) \in K^*. \end{aligned}$$

Representing y in terms of s and using the fact that K is self dual, the dual problem is equivalent to

$$\begin{aligned} \max \quad & y_1 \\ -(1 + y_1, y_2, y_2, y_1) & \in K. \end{aligned}$$

This implies that

$$(1 + y_1)^2 + y_2^2 \leq y_2^2 \quad \text{and} \quad y_1, y_2 \leq 0.$$

The first inequality implies $(1 + y_1)^2 \leq 0$, which gives $y_1 = -1$. So $(-1, 0)$ is an optimal solution to the dual problem with an objective value -1. Therefore the optimal objective of the primal problem does not equal the optimal objective of the dual. Hence strong duality does not always hold.

5 Slater Constraint Qualification

The following theorem provides a sufficient condition for strong duality in conic programming problems.

Theorem 2. *If there exists a point x in the relative interior of K such that $Ax = b$, then if the primal problem has an optimal solution, then the dual problem has an optimal solution, and the optimal values are equal, so strong duality holds.*

See [2] for a proof of this theorem.

Note that for the counterexample to strong duality above, there are no points x in the relative interior of K such that $Ax = b$, so we cannot apply the Slater constraint qualification.

References

[1] S.P. Boyd and L. Vandenberghe. *Convex Optimization.* Cambridge University Press, 2004.

[2] B. Gartner and Matousek J. Approximation algorithms and semidefinite programming: Cone programming. `http://www.ti.inf.ethz.ch/ew/lehre/ApproxSDP09/notes/conelp.pdf`.

[3] O. Güler. *Foundations of Optimization.* Graduate Texts in Mathematics. Springer New York, 2010.

[4] Andrzej Ruszczynski. *Nonlinear Optimization*. Princeton University Press, Princeton, NJ, USA, 2006.

[5] David Williamson. Orie6300: Mathematical programming I, lecture 26. `http://people.orie.cornell.edu/dpw/orie6300/Lectures/lec26.pdf`.