A Recursive Formula for the Moments of a Truncated Univariate Normal Distribution

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Let $X \sim \text{Normal}(\mu, \sigma^2)$ and $a, b \in [-\infty, \infty]$ with $a < b$. Consider now,

$$m_k := \mathbb{E}[X^k \,|\, a \le X \le b].$$

A neat recursive formula for m_k (based on $m_{-1} = 0$ and $m_0 = 1$) is:

$$\boxed{m_k = (k-1)\,\sigma^2\, m_{k-2} + \mu\, m_{k-1} - \sigma \frac{b^{k-1}\phi\left(\frac{b-\mu}{\sigma}\right) - a^{k-1}\phi\left(\frac{a-\mu}{\sigma}\right)}{\Phi\left(\frac{b-\mu}{\sigma}\right) - \Phi\left(\frac{a-\mu}{\sigma}\right)}, \qquad k = 1, 2, \ldots,}$$

here $\phi(\cdot)$ is the standard normal PDF and $\Phi(\cdot)$ is the standard normal CDF, namely,

$$\Phi(x) = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} e^{-\frac{t^2}{2}}\, dt.$$

Note the the final term in the formula vanishes if $a = -\infty$ and $b = \infty$. Applying this formula for the first few moments we obtain,

$$m_1 = \mu - \sigma \frac{\phi(\frac{b-\mu}{\sigma}) - \phi(\frac{a-\mu}{\sigma})}{\Phi(\frac{b-\mu}{\sigma}) - \Phi(\frac{a-\mu}{\sigma})},$$

$$m_2 = \mu^2 + \sigma^2 - \sigma \frac{(\mu+b)\phi(\frac{b-\mu}{\sigma}) - (\mu+a)\phi(\frac{a-\mu}{\sigma})}{\Phi(\frac{b-\mu}{\sigma}) - \Phi(\frac{a-\mu}{\sigma})},$$

$$m_3 = \mu^3 + 3\mu\sigma^2 - \sigma \frac{(\mu^2 + 2\sigma^2 + b\mu + b^2)\phi(\frac{b-\mu}{\sigma}) - (\mu^2 + 2\sigma^2 + a\mu + a^2)\phi(\frac{a-\mu}{\sigma})}{\Phi(\frac{b-\mu}{\sigma}) - \Phi(\frac{a-\mu}{\sigma})},$$

$$m_4 = \mu^4 + 6\mu^2\sigma^2 + 3\sigma^4 - \sigma \frac{(b^3 + b^2\mu + b\mu^2 + \sigma^2(3b + 5\mu) + \mu^3)\phi(\frac{b-\mu}{\sigma}) - (a^3 + a^2\mu + a\mu^2 + \sigma^2(3a + 5\mu) + \mu^3)\phi(\frac{a-\mu}{\sigma})}{\Phi(\frac{b-\mu}{\sigma}) - \Phi(\frac{a-\mu}{\sigma})}.$$

Derivation:
Define, $n_0 := \mathbb{P}(a \le X \le b)$. So, $n_0 = \Phi(\frac{b-\mu}{\sigma}) - \Phi(\frac{a-\mu}{\sigma})$. Now,

$$\begin{aligned}
m_k n_0 &= \int_a^b x^k \, \frac{1}{\sigma}\phi\Big(\frac{x-\mu}{\sigma}\Big)\,dx \\
&= \int_a^b \frac{x^k}{\sigma\sqrt{2\pi}}\, e^{-\frac{(x-\mu)^2}{2\sigma^2}}\,dx \\
&= \int_a^b \frac{\sigma\, x^{k-1}}{\sqrt{2\pi}}\,\Big(\frac{x-\mu}{\sigma^2} + \frac{\mu}{\sigma^2}\Big)\, e^{-\frac{(x-\mu)^2}{2\sigma^2}}\,dx \\
&= \sigma \int_a^b \frac{-x^{k-1}}{\sqrt{2\pi}}\,\frac{-(x-\mu)}{\sigma^2}\, e^{-\frac{(x-\mu)^2}{2\sigma}}\,dx \,+\, \int_a^b \mu\,\frac{x^{k-1}}{\sigma\sqrt{2\pi}}\, e^{-\frac{(x-\mu)^2}{2\sigma^2}}\,dx \\
&= \sigma \Big[\frac{-x^{k-1}}{\sqrt{2\pi}}\, e^{-\frac{(x-\mu)^2}{2\sigma^2}}\Big]_a^b + \int_a^b (k-1)\sigma\,\frac{x^{k-2}}{\sqrt{2\pi}}\, e^{-\frac{(x-\mu)^2}{2\sigma^2}}\,dx \,+\, \mu \int_a^b x^{k-1}\,\frac{1}{\sigma}\phi\Big(\frac{x-\mu}{\sigma}\Big)\,dx \\
&= \sigma\Big[-b^{k-1}\phi\Big(\frac{b-\mu}{\sigma}\Big) + a^{k-1}\phi\Big(\frac{a-\mu}{\sigma}\Big)\Big] \,+\, (k-1)\sigma^2 \int_a^b x^{k-2}\,\frac{1}{\sigma}\phi\Big(\frac{x-\mu}{\sigma}\Big)\,dx + \mu \int_a^b x^{k-1}\,\frac{1}{\sigma}\phi\Big(\frac{x-\mu}{\sigma}\Big)\,dx.
\end{aligned}$$

Hence,

$$m_k = -\sigma\,\frac{b^{k-1}\phi\Big(\frac{b-\mu}{\sigma}\Big) - a^{k-1}\phi\Big(\frac{a-\mu}{\sigma}\Big)}{n_0} \,+\, (k-1)\sigma^2 \int_a^b x^{k-2}\,\frac{1}{n_0\sigma}\phi\Big(\frac{x-\mu}{\sigma}\Big)\,dx + \mu \int_a^b x^{k-1}\,\frac{1}{n_0\sigma}\phi\Big(\frac{x-\mu}{\sigma}\Big)\,dx.$$

Observing that the integrals in the second and third terms are lower order moments and reorganizing, we obtain the result.