Dynamics of an Abstract Production System

Bachelor Final Project

Selen, J. 0637922

Supervisor Prof.dr.ir. Rooda, J.E.
Coaches Dr.ir. Lefeber, A.A.J.
Dr. Nazarathy, J.

Eindhoven University of Technology
Department of Mechanical Engineering
Systems Engineering Group

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Abstract

In this paper, a cyclic push-pull system is analyzed. The analysis focuses on the influence of a pull-first policy on (asymptotical) stability, for both general and specific cases. Such an analysis is useful as the cyclic push-pull system allows production of an arbitrary number of products, it allows full utilization of all machines and it is often stable.

With the use of steady state flow equations an expression governing the stability of a system is derived. The stability of a system depends on its push- and pull-parameters.

We give a complete characterization of systems with two, three or four machines. For a combination of the push- and pull-parameters a cyclic behaviour is witnessed, which, dependent of the number of machines, is either permanent or transient. A condition for asymptotical stability is given for systems with an odd number of machines.

A system with four machines and a certain combination of push- and pull-parameters has a transient cyclic behaviour which is analyzed in greater detail. Another case is analyzed for this system with a different combination of push- and pull-parameters. This system can be either asymptotically stable or only stable, a number of restrictions on these parameters are derived that make the system asymptotically stable.

Contents

Chapter 1 Introduction and model

Manufacturing systems or production systems exist in a wide variety. A production system is a network of machines in which processes occur. The objective of such a network is to generate products. As the complexity of these systems increase rapidly over the last decade, the costs involved also go up tremendously. Therefore, preventing failures and optimizing the network is important. Before optimization can take place, parts of the network have to be modelled to gain insight into the behaviour of the system.

This paper focuses on the analysis of an abstract production system. Such an analysis is important because it allows production of an arbitrary number of products. This model also allows full utilization and it is often stable.

The model of the system is explained in full in the next section. A definition of stability is given and an overview of all the results concludes the first chapter. In Chapter 2 general properties concerning the (asymptotical) stability of a system are derived. Chapter 3 uses these general properties to give a complete characterization of systems with either two, three of four machines. A final open question is answered with the use of simulation in Chapter 4. In [1], [2] and [4] this type of system with two machines is analyzed in great detail.

1.1 Cyclic push-pull system

Consider an abstraction of a production system. M machines numbered $1,...,M$ produce M types of products numbered $1,...,M$. Each product undergoes two steps. Products of type i are first created at machine i and then moved to machine $i+1$ (in case $i=M$, interpret $i+1$ as 1) Note that the system is cyclic and that therefore all index arithmetic (i) is modular M. At machine $i+1$, products of type i are further processed and then leave the system. Thus each machine (i) can perform two types of operations: *creation* of products of type i and *processing* of products of type $i-1$. The product that is processed leaves the system.

Products queue at the machines while waiting for their processing step. The system contains M queues, labelled $X_1,...,X_M$. Queue X_i is for products of type i and is next to machine $i+1$.

The machines need to divide their time between creating and processing. The policy that is analysed in this paper is the pull-first policy in which machines give pre-emptive priority to processing. This means that when ever X_{i-1} is not empty, machine i will work on processing. Otherwise, the machine works on creating and is stopped at the instant in which X_{i-1} becomes non-empty. Thus, machines never idle.

As an approximation, the products are a continuous quantity. Each product i is associated with positive production rates λ_i and μ_i. The variable λ_i is the rate at which product i is created at machine i. The creation of products with rate λ_i is a simplification of a process in which products are processed at rate λ_i from an infinite queue outside the system. μ_i is the rate at which product i is processed from queue X_i at machine $i+1$. Denote by $X_i(t)$ the continuous non-negative quantity of material at queue X_i at time t.

After products are processed, they leave the system. The long term average output rate of product i is v_i.

Machines spent time on both processing and creating products, θ_i is the long term fraction of time spent on creating product i. A machine never idles, therefore $1-\theta_i$ is the long term fraction of time spent on processing product $i-1$.

An illustration of the system with $M=3$ is presented in Figure 1.1.

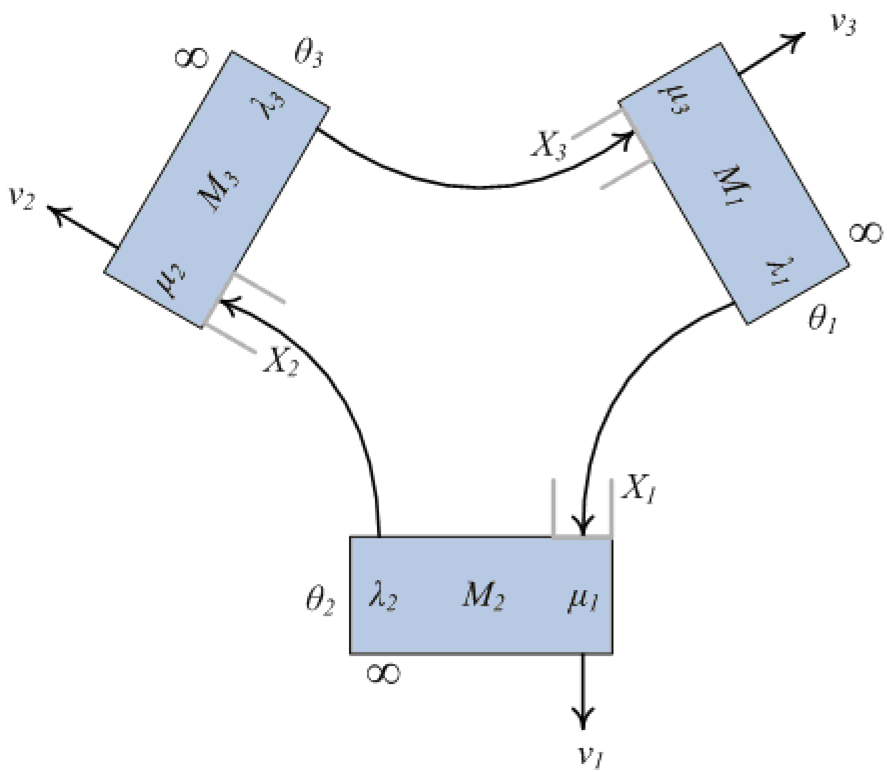

Figure 1.1 – A cyclic push-pull system with three machines.

The system behaves differently for every combination of empty and non-empty queues, i.e. if all queues are non-empty, the machines work on processing the products, if one queue empties, one machine will start creating products. A combination of empty and non-empty queues is defined as a mode and is noted in binary digits, e.g. mode $\{1,0,0\}$ is the mode in which queue X_1 is non-empty and X_2 and X_3 are empty. The mode number is defined as the decimal representation of the binary mode plus 1. Thus, in general, the modes are $i=1,...,2^M$. In each mode, M ordinary differential equations (ODEs) govern the behaviour of $X(t)$.

The time needed to empty a queue when a machine $i+1$ fully works on processing product i, i.e. there is no creation of product i, is $Y_i = X_i / \mu_i$. This variable is used throughout the analysis instead of X_i.

The governing variable of the system is ρ_i defined as $\rho_i = \lambda_i / \mu_i$. Two ranges can be distinguished, either $0 < \rho_i < 1$ or $\rho_i > 1$. For the first range , $\lambda_i < \mu_i$ and the machines process product i faster than it can be created. For the second range $\lambda_i > \mu_i$ and product i can be created faster than it is processed. The singular case, $\rho_i = 1$ is not taken into account in this paper in most cases. The behaviour of the system depends on ρ_i.

1.2 Definition of stability

A general goal is to understand under which conditions a cyclic push-pull system may operate in a stable manner. Before being able to analyze stability, one has to define the term stability, from [3].

For a system $\dot{A} = f(A,t)$ with $f(0,t) = 0,\ \forall t \geq 0$, the equilibrium point $A = 0$ is

- *stable* if, for each $\varepsilon > 0$, there is $\delta = \delta(\varepsilon, t_0) > 0$ such that

$$\|A(t_0)\| < \delta \Rightarrow \|A(t)\| < \varepsilon, \quad \forall t \geq t_0 \geq 0 \tag{1.1}$$

  Figure 1.2 is a graphical visualization of (1.1).

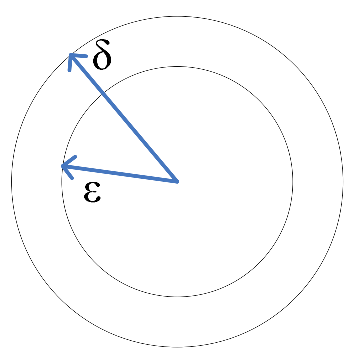

Figure 1.2 – Definition of stability.

- *asymptotically stable* if it is stable and there is $c = c(t_0) > 0$ such that $A(t) \to 0$ as $t \to \infty$, for all $\|A(t_0)\| < c$.

This definition can be translated to the cyclic push-pull system. A system is stable if all θ_i are feasible, i.e. all θ_i have a unique solution and are within 0 and 1. A negative or

$\theta_i > 1$ is a (mathematically) possible outcome of the steady state flow equations (presented in section 2.2), however, it is not a feasible outcome.

A system is asymptotically stable if it is stable, and $X(t) \to 0$ as $t \to \infty$ starting from anywhere in the state space.

1.3 Overview of results

In this section all the results obtained throughout the paper are presented. First, the general properties of a system with M machines are given. Secondly, the characterization of systems with two, three or four machines is shown. Lastly, the permanent cyclic behaviour for arbitrary M, tackled in Chapter 4, is presented.

General properties concerning stability

If we assume a system is in an equilibrium point and it is stable, the queues do not increase over time. Therefore, conservation of mass can be applied and the input and output rate of each product i are equal. The fraction of time spent on creating product i, θ_i, multiplied with the creation rate λ_i is the input rate of product i. The output rate is described by the product of the fraction of time spent on processing of product i, $1-\theta_{i+1}$, and the processing rate μ_i. With the help of $\rho_i = \lambda_i / \mu_i$, we obtain

$$\rho_i \theta_i + \theta_{i+1} = 1$$

For a system with M machines, there are M of the above equations. The above equation can be solved for θ_i. A generalized expression for θ_i and arbitrary M is presented below.

$$\theta_i = \frac{\sum_{j=1}^{M-1} \left[(-1)^{j+1} \prod_{k=j}^{M-1} \rho_{i+k} \right] + (-1)^{M+1}}{\prod_{l=1}^{M} \rho_l + (-1)^{M+1}}$$

For a system consisting of M machines and all $\rho_i < 1$, the system is always asymptotically stable and converges to the mode in which all queues are empty. The machines can divide their time between creating and processing products while still maintaining steady queues.

System with two machines

For a system with two machines, there are two regions that are stable, only one region is also asymptotically stable, see Figure 1.3.

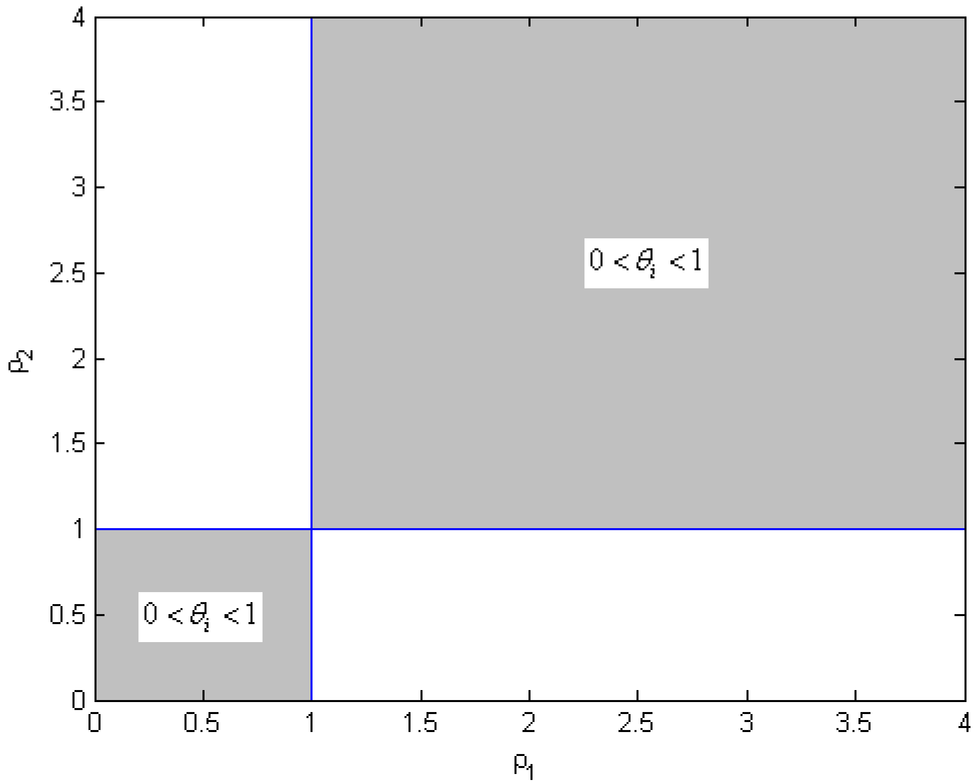

Figure 1.3 – Regions of stability in the state space of a system with two machines

System with three machines

There are four regions that, with the use of some restrictions, are stable for a system with three machines, these regions are

1. $\rho_i < 1$
2. $\rho_3 > 1, \rho_{1,2} < 1$
3. $\rho_1 < 1, \rho_{2,3} > 1$
4. $\rho_i > 1$

The first three regions are always asymptotically stable. If all $\rho_i > 1$ the system is asymptotically stable when $\prod_{i=1}^{3} (\rho_i - 1) < 1$. It is asymptotically unstable when $\prod_{i=1}^{3} (\rho_i - 1) > 1$ and marginally stable for $\prod_{i=1}^{3} (\rho_i - 1) = 1$. Figure 1.4 clearly shows the difference, both started with the same initial condition, close to $\underline{X}(0) = \{2,1,2\}$. The figure on the right increases over time, whereas the figure on the left continually decreases.

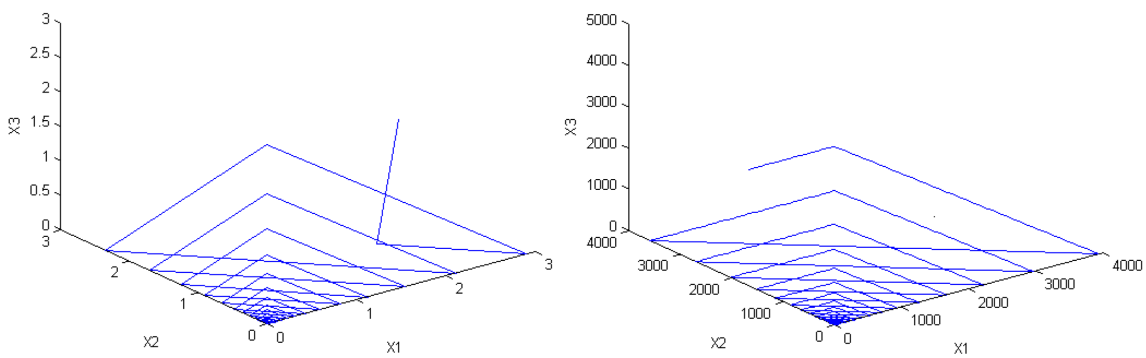

Figure 1.4 – Trajectories for $\rho_i > 1$, for the left figure $\prod_{i=1}^{3}(\rho_i - 1) < 1$, and for the right $\prod_{i=1}^{3}(\rho_i - 1) > 1$

System with four machines

Six cases can be distinguished for $M = 4$, these are

1. $\rho_i < 1$
2. $\rho_1 > 1, \rho_{2,3,4} < 1$
3. $\rho_{1,2} > 1, \rho_{3,4} < 1$
4. $\rho_{1,3} > 1, \rho_{2,4} < 1$
5. $\rho_{1,2,3} > 1, \rho_4 < 1$
6. $\rho_i > 1$

In which case 4 is always unstable and case 3 is divided into two cases. Case 1, 2, the first case of case 3 and case 5 are asymptotically stable. For the second case of case 3 some parts of the state space are asymptotically stable whereas the rest is not. There are five routes that lead to an asymptotically stable system, for each of these routes a restriction is placed on the parameters in mode 16, which is the starting mode. These restrictions are

$$\left\{(Y_1^{16}, Y_2^{16}, Y_3^{16}, Y_4^{16}) \middle| Y_1^{16} < Y_2^{16} \wedge Y_1^{16} < Y_3^{16} < Y_4^{16}\right\}$$

$$\left\{(Y_1^{16}, Y_2^{16}, Y_3^{16}, Y_4^{16}) \middle| Y_2^{16} < Y_1^{16} < Y_3^{16} + \rho_3(Y_1^{16} - Y_2^{16}) < Y_4^{16}\right\}$$

$$\left\{(Y_1^{16}, Y_2^{16}, Y_3^{16}, Y_4^{16}) \middle| Y_2^{16} < Y_3^{16} < Y_1^{16} < \frac{1}{1-(1-\rho_3)\rho_4}\left[Y_4^{16} + \rho_4 Y_3^{16} + \rho_3\rho_4 Y_2^{16}\right]\right\}$$

$$\left\{(Y_1^{16}, Y_2^{16}, Y_3^{16}, Y_4^{16}) \middle| Y_3^{16} < Y_2^{16} < Y_1^{16} < \frac{Y_4^{16} - \rho_3\rho_4 Y_2^{16} + (2\rho_3 - 1)\rho_4)Y_3^{16}}{1-(1-\rho_3)\rho_4}\right\}$$

$$\left\{(Y_1^{16}, Y_2^{16}, Y_3^{16}, Y_4^{16}) \middle| Y_3^{16} < Y_1^{16} < Y_2^{16} \wedge Y_1^{16} < \frac{1}{1-\rho_4}(Y_4^{16} - \rho_4 Y_3^{16})\right\}$$

All of these restrictions span a 4D volume and therefore the system can indeed be asymptotically stable if it meets any of the above five restrictions.

In the sixth case a transient cyclic behaviour is witnessed. This behaviour is analyzed in two stages. To be able to stay within the cycle, starting in mode 15, the following condition must hold

$$\left[\frac{\rho_1 - 1}{\rho_4} + 1\right] Y_3^{15} < Y_2^{15} < \left[\rho_1 - \frac{(\rho_1 - 1)(\rho_4 - 1)}{\rho_3 + \rho_4 - 1}\right] Y_3^{15}$$

If the system can stay within the cycle, it does not mean the system is also asymptotically stable. For it to be asymptotically stable, the following restriction has to be valid as well

$$-2 + \rho_1\rho_3 + \rho_2\rho_4 < (\rho_1 - 1)(\rho_2 - 1)(\rho_3 - 1)(\rho_4 - 1) < 1$$

The latter of the two cannot hold and therefore the system is not asymptotically stable for any parameter setting. If it can stay in the cycle the queues will always increase over time.

Asymptotic stability for a permanent cyclic behaviour

A permanent cyclic behaviour is witnessed for $M = 2K + 1$, $K = 1, 2, 3, ...$ and all $\rho_i > 1$.

By means of simulation, the critical value for which the system becomes marginally stable is sought. All ρ_i are equal and are labelled $\overline{\rho}$. The variable $\overline{\rho}_K$ is this critical value, for which the system is asymptotically stable.

The simulation is done with the help of the M-files presented in Appendix A, for $K = 1, 2, 3, 4, 5, 6, 20$. The results are plotted in Figure 4.1.

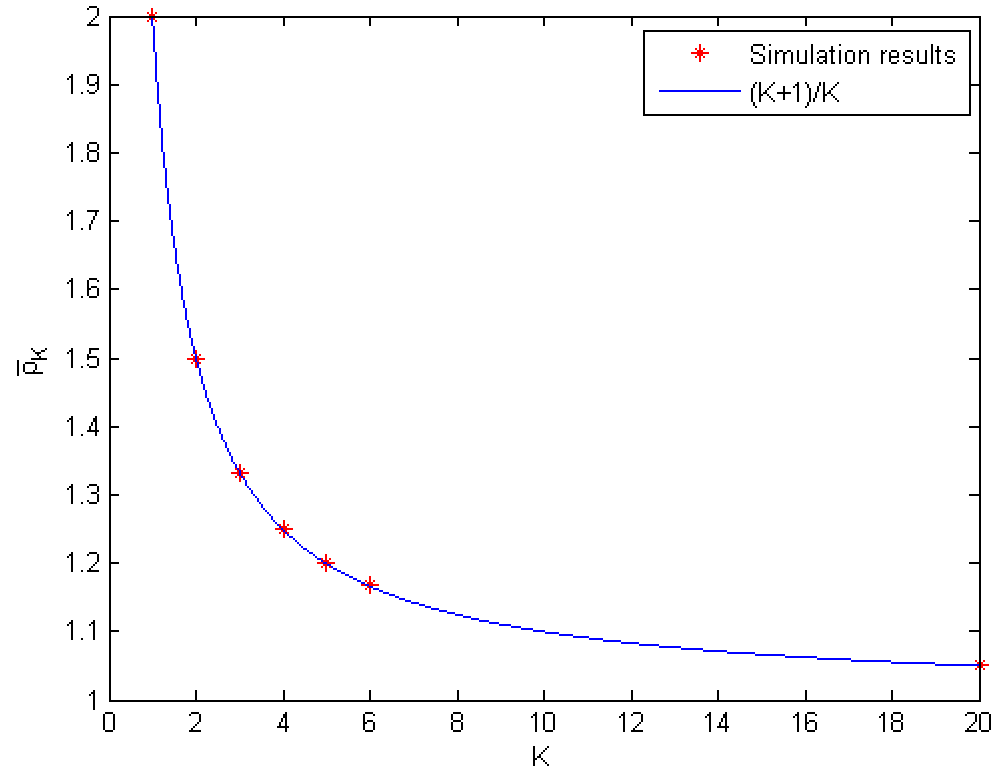

Figure 1.5 – Simulation results and fitted curve.

It can be seen that the fitted curve $(K+1)/K$ corresponds with the simulation results. All systems with $M = 2K+1$ and $(K+1)/K > \overline{\rho} > 1$ behave asymptotically stable.

Chapter 2 General properties concerning stability

This chapter focuses on generalizing properties of a system of M machines concerning (asymptotical) stability. The chapter is divided in two parts.

In the first part the steady state flow equations for a system of M machines are derived. This is done by first looking at a system with $M=2$ and then generalizing these equations for arbitrary M. For this analysis, it is assumed that the system is stable. The derived set of equations can be used to identify stability in systems for all M.

The second part examines a system for which all $\rho_i < 1$. For this system an analysis on asymptotical stability is done. The conclusions for this case are generalized for all M.

All variables that are used are explained in section 1.1.

2.1 Illustrative example for two machines

This section concerns the steady state flow equations for a system with two machines and an analysis of stability is done. Figure 2.1 shows the push-pull system.

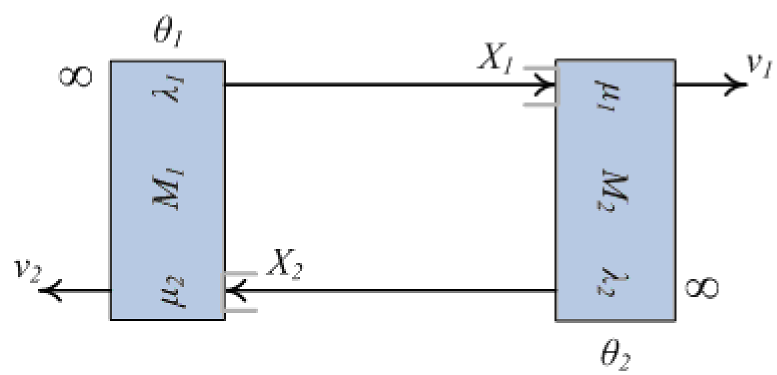

Figure 2.1 – A cyclic push-pull system with two machines.

If the system is in an equilibrium point and it is stable, the queues do not increase over time, see definition (1.1). Therefore, conservation of mass can be applied and the input and output rate of each product i are equal. The fraction of time spent on creating product 1, θ_1, multiplied with the creation rate λ_1 is the input rate of product 1. The output rate is described by the product of the fraction of time spent on processing of product 1, $1-\theta_2$, and the processing rate μ_1. In a stable system, both the input and output rate are equal to the long term average output rate of product i, v_1. The same applies for the second product, this gives the following two relations.

$$\begin{aligned} \theta_1\lambda_1 &= v_1 = (1-\theta_2)\mu_1 \\ \theta_2\lambda_2 &= v_2 = (1-\theta_1)\mu_2 \end{aligned} \tag{2.1}$$

If (2.1) is rewritten using the definition $\rho_i = \lambda_i / \mu_i$, we obtain

$$\begin{aligned} \rho_1\theta_1 + \theta_2 &= 1 \\ \rho_2\theta_2 + \theta_1 &= 1 \end{aligned} \tag{2.2}$$

which can be solved for θ_i

$$\begin{aligned} \theta_1 &= \frac{\rho_2 - 1}{\rho_1\rho_2 - 1} \\ \theta_2 &= \frac{\rho_1 - 1}{\rho_1\rho_2 - 1} \end{aligned} \tag{2.3}$$

The steady state flow equations for $M = 2$ and the expressions for θ_1 and θ_2 are derived under the assumption that the system is stable. Therefore, for a system to be stable, θ_i must be feasible, i.e. it has a unique solution and $0 < \theta_i < 1$ has to hold. This holds for arbitrary M. The expressions for θ_i for $M = 2$ are used in section 3.1 to analyze stability.

2.2 Steady state flow equations for arbitrary M

Assume a system of M machines, Figure 2.2 shows two of the machines in this system.

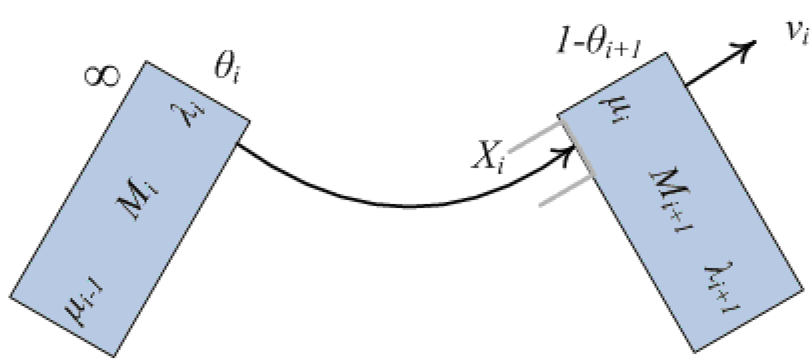

Figure 2.2 – Two machines that are part of a system consisting of M machines.

An identical approach is used to derive the steady state flow equations for arbitrary M. Assume the system is in an equilibrium point and it is stable, queues do not increase over time. Therefore, conservation of mass can be applied and the input and output rate of each product i are equal. The fraction of time spent on creating product i, θ_i, multiplied with the creation rate λ_i is the input rate of product i. The output rate is described by the product of the fraction of time spent on processing of product i, $1-\theta_{i+1}$, and the processing rate μ_i. In a stable system, both the input and output rate are equal to the long term average output rate of product i, ν_i.

$$\theta_i \lambda_i = \nu_i = (1-\theta_{i+1})\mu_i \tag{2.4}$$

For a system of M machines, there are M of the above equations. If (2.4) is rewritten using the definition $\rho_i = \lambda_i / \mu_i$, we obtain

$$\rho_i \theta_i + \theta_{i+1} = 1 \tag{2.5}$$

This set of equations can be rewritten in the form

$$\begin{pmatrix} \rho_1 & 1 & 0 & \cdots & \cdots & 0 \\ 0 & \rho_2 & 1 & 0 & \cdots & \vdots \\ \vdots & 0 & \ddots & \ddots & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & \vdots & & \ddots & \ddots & 1 \\ 1 & 0 & \cdots & \cdots & 0 & \rho_M \end{pmatrix} \begin{pmatrix} \theta_1 \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ \theta_M \end{pmatrix} = \begin{pmatrix} 1 \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ 1 \end{pmatrix} \tag{2.6}$$

The system is solved for θ_i with a left matrix divide. An M-file is made to solve this set of equations for arbitrary M and ρ, see Appendix A.

The output of the M-file is $\theta_1,...,\theta_M$ as a function of ρ_i. Using this M-file a general expression θ_i for $M = 5$ can be derived.

$$\theta_i = \frac{\rho_{i+1}\rho_{i+2}\rho_{i+3}\rho_{i+4} - \rho_{i+2}\rho_{i+3}\rho_{i+4} + \rho_{i+3}\rho_{i+4} - \rho_{i+4} + 1}{\rho_1\rho_2\rho_3\rho_4\rho_5 + 1} \tag{2.7}$$

Note that the system is cyclic and indices 'loop around', i.e. for $i = 4$ the expression $i+3$ is 2 instead of 7. A similar expression for $M = 6$ is made.

$$\theta_i = \frac{\rho_{i+1}\rho_{i+2}\rho_{i+3}\rho_{i+4}\rho_{i+5} - \rho_{i+2}\rho_{i+3}\rho_{i+4}\rho_{i+5} + \rho_{i+3}\rho_{i+4}\rho_{i+5} - \rho_{i+4}\rho_{i+5} + \rho_{i+5} - 1}{\rho_1\rho_2\rho_3\rho_4\rho_5\rho_6 - 1} \tag{2.8}$$

A pattern can be found between (2.7) and (2.8). The numerator of (2.7) is multiplied by ρ_{i+5} and 1 is subtracted. When looking at more θ_i for a number of M it can be seen that there is a combination of a summation and a product operator. The denominator is a summation of the product of all ρ_i and either 1 or -1. A generalized expression for θ_i for arbitrary M can be seen below.

$$\theta_i = \frac{\sum_{j=1}^{M-1}\left[(-1)^{j+1}\prod_{k=j}^{M-1}\rho_{i+k}\right]+(-1)^{M+1}}{\prod_{l=1}^{M}\rho_l+(-1)^{M+1}} \tag{2.9}$$

By entering (2.9) in (2.5) for i and $i+1$ the expression can be checked for correctness.

In these two sections the steady state flow equations were derived for $M=2$ and arbitrary M. The variable θ_i governs the stability of the system, the solution for θ_i has to be feasible. Also, a general expression for θ_i is computed, which can be seen in (2.9). In the next section asymptotical stability for a system of M machines and $\rho_i<1$ is analyzed.

In Chapter 3 the stability of systems with two, three and four machines are analyzed using the steady state flow equations and θ_i. The asymptotical stability is examined only if the system is stable.

2.3 Asymptotical stability for all M and $\rho_i<1$

In this section proof is provided that for a system consisting of M machines and all $\rho_i<1$ the system will always be asymptotically stable with the used policy and is therefore also stable. First, stability is analyzed, subsequently, the asymptotical stability is examined.

If the variable $\rho_i=\lambda_i/\mu_i$ is smaller than 1 for all i, the rate of creation λ_i is smaller than the rate of processing μ_i. This means that a queue i will not increase if both the push- and pull-activities for that same product i are performed at full capacity. This indicates that, if the system is in an equilibrium point, all θ_i are within 0 and 1. Each machine can divide its time between creating and processing so that $\dot{Y}=0$. A simulation is done for three to twenty machines each with uniform random values for ρ_i between 0 and 1. Only the first three θ_i are plotted, see Figure 2.3. The simulation confirms the presumption that the system is always stable if all $\rho_i<1$.

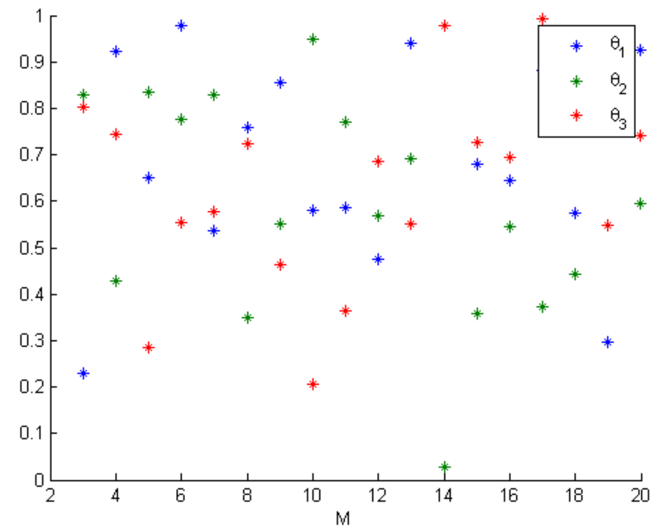

Figure 2.3 – Values for the first three θ_i for systems with all $\rho_i < 1$

Assume the initial queues are all non-empty and therefore the mode is $\{1,...,1\}$. With the pull-first policy the machine will always work on processing if the queue is non-empty. The ODEs prescribe a decrease in all queues, see Appendices B and C for an example of the ODEs per mode. This decrease continues until one of the queues empties. This changes the mode to e.g. $\{1,...,1,0,1,...,1\}$. In this mode, and actually in any mode outside $\{0,...,0\}$, the ODEs are all negative or zero. If machine $i-1$ starts creating products, machine i responds by starting to process these products. Either an equilibrium ($\dot{Y}_i = 0$) is reached or the ODEs are negative, $\dot{Y}_i < 0$. Because all ODEs are negative or zero, the system will continue to diverge to mode $\{0,...,0\}$ and will reach it for $t \to \infty$. This is in accordance with the definition of asymptotical stability from section 1.2. Figure 2.4 shows two trajectories for two and three machines respectively, both with $\rho_i < 1$. It can be seen that the system converges to the mode in which all queues are non-empty. The trajectories are generated with an M-file that can be seen in Appendix A.

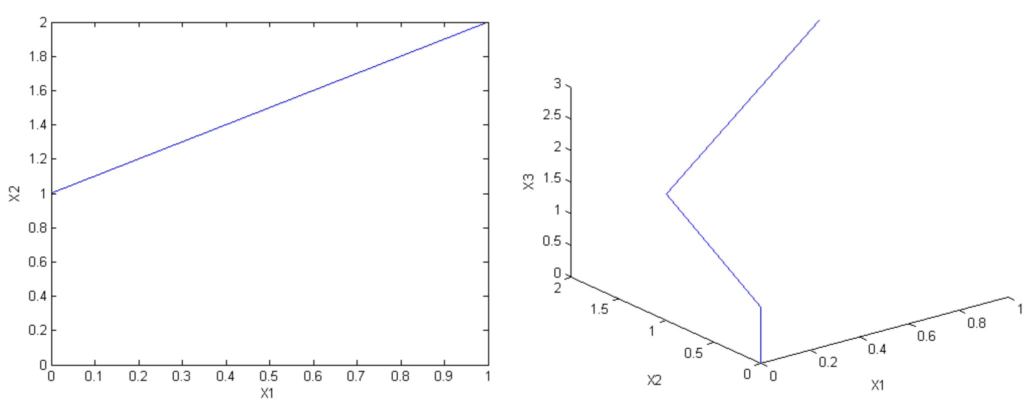

Figure 2.4 – Trajectories for $M = 2$ and $M = 3$ with all $\rho_i < 1$.

Chapter 3 Complete characterization of M = 2, 3 and 4

In this chapter systems with either two, three or four machines are analyzed. First, the systems are split into a number of cases (combinations of ρ_i). These cases have to be stable before an analysis of asymptotical stability can be made. If a case is unstable it is not analyzed. When possible, a figure is made showing the regions in the state space for which the system is stable.

The definition of (asymptotical) stability can be found in section 1.2. The variables that are used are first introduced and explained in section 1.1.

3.1 Analysis of a system with two machines

In this section a push-pull system consisting of two machines is analyzed. There are three cases that describe the system. First, for these cases the stability is examined. Only for the stable cases asymptotical stability is studied, this is done by writing down the ODEs for each mode. Modes and ODEs are explained in section 1.1.

The stability of the system depends on ρ_i, the steady state flow equations are derived in section 2.1 and an expression for θ_1 and θ_2 is shown in (2.3) and copied once more below.

$$\begin{aligned}\theta_1 &= \frac{\rho_2 - 1}{\rho_1\rho_2 - 1}\\ \theta_2 &= \frac{\rho_1 - 1}{\rho_1\rho_2 - 1}\end{aligned} \tag{3.1}$$

As was already stated in section 1.1, there are two regions for ρ_i that are analyzed, either $\rho_i < 1$ or $\rho_i > 1$. The singular case in which $\rho_i = 1$ is not taken into account, θ_i does not have a unique solution and is therefore not a feasible outcome.

There are three cases (combinations) of ρ_i,

1. $\rho_{1,2} > 1$
2. $\rho_1 > 1, \rho_2 < 1$, this is the same as $\rho_1 < 1, \rho_2 > 1$ because it is a cyclic system.
3. $\rho_{1,2} < 1$

It can be seen that for case 2, using (3.1), the system is unstable. The denominator $\rho_1\rho_2 - 1$ in (3.1) will either be positive or negative and the sign of the numerator of θ_1 is opposite of the one of θ_2, so one of the two θ_i is negative. This is physically impossible and therefore the solution is not feasible and the system is unstable.

For both case 1 and 3 θ_i is positive. In case 1, the denominator $\rho_1\rho_2 - 1$ is positive and the numerator $\rho_i - 1$ is also positive. For case 3, the denominator is always negative, the numerator is negative as well. The third case has been analyzed in section 2.3.

When looking at (2.2), and with ρ_i positive, $0 < \theta_i < 1$ has to hold if all $\theta_i > 0$. This means that the system behaves in a stable manner in an equilibrium point.

In the state space, these unstable and stable regions can be visualised, see Figure 3.1.

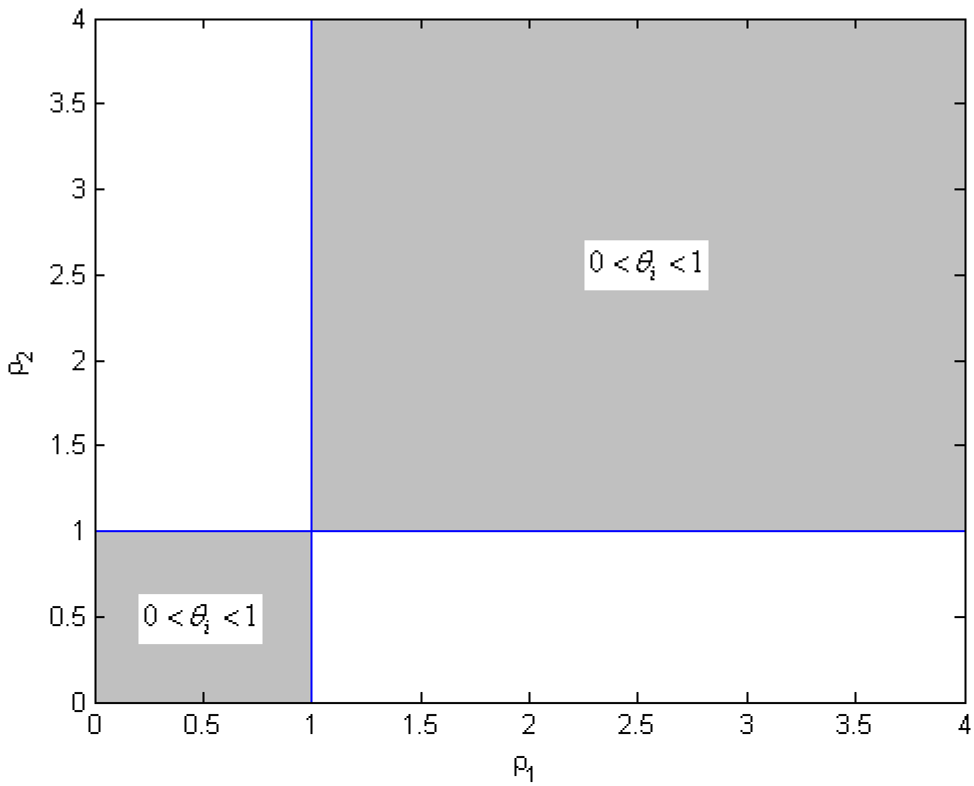

Figure 3.1 – Regions of stability in the state space of a system with two machines

Now that the analysis of stability is complete, case 1 can be examined for asymptotical stability.

The ODEs for each mode are derived, there are $2^2 = 4$ modes. In each mode, 2 ODEs govern the behaviour of $Y(t)$.

Case 1 : $\rho_{1,2} > 1$

The ODEs for the four modes are given below. Mode 2 and 3 are identical, only the indices are reversed, therefore mode 3 is not displayed.

Mode 1 - $\{0,0\}$

$$\dot{Y}_1 = \theta_1\rho_1 - (1-\theta_2) = 0$$

$$\dot{Y}_2 = \theta_2\rho_2 - (1-\theta_1) = 0$$

Mode 2 - $\{0,1\}$

$$\dot{Y}_1 = 0$$

$$\dot{Y}_2 = \rho_2 - 1 > 0$$

Mode 4 - $\{1,1\}$

$$\dot{Y}_1 = -1 < 0$$

$$\dot{Y}_2 = -1 < 0$$

Modes 2, 3 and 4 are called absorbing modes, these are modes from which, when entered, the system cannot leave. While the system is stable, it is not asymptotically stable.

Assume a system of two machines with non-empty queues as an initial condition, from this mode, only modes 2 and 3 can be reached. However, there is a possibility that mode 1 is reached when $Y_1 = Y_2$, in the state space, this is represented by a line in a 2D plane. The relative volume of this line can be neglected and therefore such a transition is not taken into account. For systems with larger M the same applies and is therefore neglected as well.

This concludes the analysis of a system with two machines, there are two regions for which the system is stable of which only one region, $\rho_{1,2} < 1$, is asymptotically stable. The next section concerns the analysis of a system with three machines.

3.2 Analysis of a system with three machines

A similar analysis as presented in section 3.1 is used to analyze a cyclic push-pull system with three machines.

First, the expressions for θ_i are derived and a number of cases are presented with different combinations of ρ_i. For these cases the ODEs are derived and a modes graph is presented for each of these cases. A modes graph is a graphical representation of how the push-pull system can change modes.

In this section a new phenomenon, a permanent cyclic behaviour, is encountered. The system keeps on changing modes without ever reaching the first mode $\{0,0,0\}$. It is analyzed if this behaviour can be asymptotically stable.

Using the steady state flow equations, three equations can be derived by using the conservation of mass, solving these equations obtains the expressions for θ_i.

$$\theta_1 = \frac{1-(1-\rho_2)\rho_3}{1+\rho_1\rho_2\rho_3},\ \theta_1 > 0\ iff\ 1-(1-\rho_2)\rho_3 > 0$$
$$\theta_2 = \frac{1-(1-\rho_3)\rho_1}{1+\rho_1\rho_2\rho_3},\ \theta_2 > 0\ iff\ 1-(1-\rho_3)\rho_1 > 0 \qquad (3.2)$$
$$\theta_3 = \frac{1-(1-\rho_1)\rho_2}{1+\rho_1\rho_2\rho_3},\ \theta_3 > 0\ iff\ 1-(1-\rho_1)\rho_2 > 0$$

The denominator is always positive and therefore the numerator has to be positive to be able to get a stable system.

Now, the stability for all the combinations of ρ_i is analyzed using the expressions derived earlier in (3.2).

The ODEs for $M = 3$ depend on the push and pull parameters μ_i and λ_i, or $\rho_i = \lambda_i / \mu_i$. There are $2^3 = 8$ different cases, each with $2^3 = 8$ modes. First, the cases will be looked at and afterwards the eight modes with corresponding ODEs for each case are given. For the stable cases, it is investigated if the system is asymptotically stable. For these cases a modes graph is made.

ρ_i is always positive and can be either $\rho_i < 1$ or $\rho_i > 1$, the singular $\rho_i = 1$ is not taken into account. This gives eight cases in total. Some of these modes are the same and without loss of generality we distinguish 4 cases.

1. $\rho_i < 1$
2. $\rho_3 > 1, \rho_{1,2} < 1$, this is the same as $\rho_1 > 1, \rho_{2,3} < 1$ and $\rho_2 > 1, \rho_{1,3} < 1$.
3. $\rho_1 < 1, \rho_{2,3} > 1$, this is the same as $\rho_2 < 1, \rho_{1,3} > 1$ and $\rho_3 < 1, \rho_{1,2} > 1$.
4. $\rho_i > 1$

Case 1 in which $\rho_i < 1$ is always stable and also asymptotically stable, the proof for stability can be found in Appendix D. This case has been analyzed for arbitrary M in section 2.3.

For the remaining cases, it is first determined if all θ_i are positive with the help of (3.2). If not all $0 < \theta_i < 1$, the system will not behave in a stable manner. The singular case, in which $\theta_i = 0$ or $\theta_i = 1$, is not stable either and is therefore not analyzed.

As a reminder, the equations derived with the conservation of mass are only valid when the system is in an equilibrium point. So if no products are created by machine 1 ($\theta_1 = 0$), queue X_1 remains unchanged and machine 2 is described by $\theta_2 = 1$. It follows from the flow equations that $\theta_3 = 1/\rho_3 = 1 - \rho_2$, this cannot hold, see Figure 3.2. Because it is a cyclic system, the same logic applies to $\theta_i = 1$, which is unstable as well. From now on, these singular cases are not analyzed for any M.

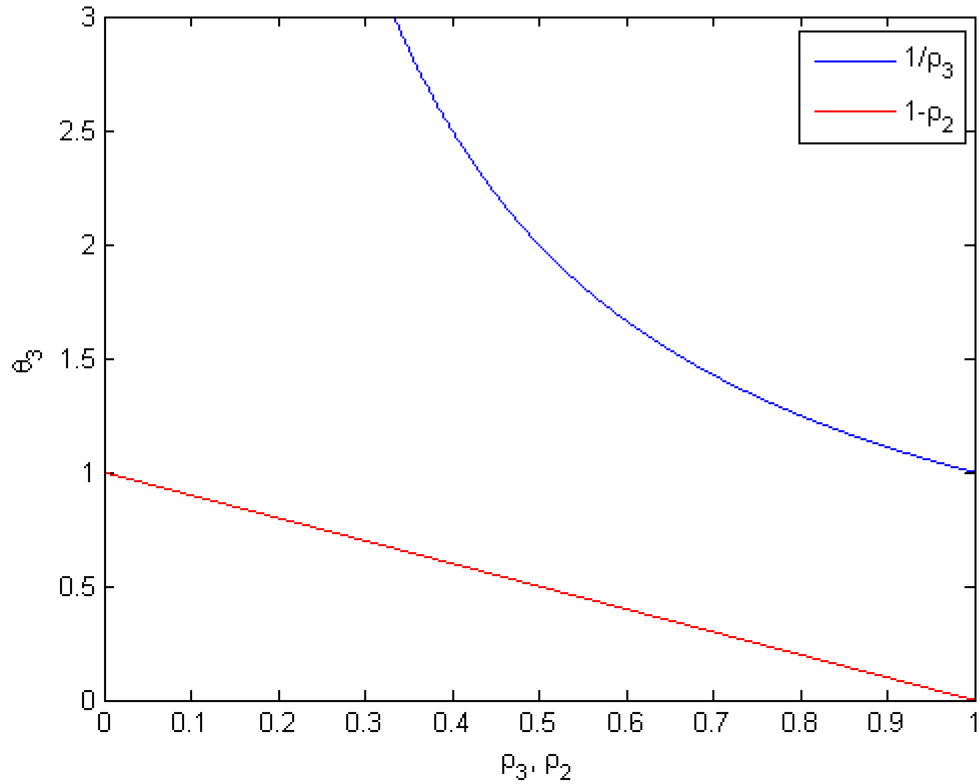

Figure 3.2 – Proof that $\theta_i = 0$ is unstable.

The ODEs for all four cases can be found in Appendix B.

Case 2 : $\rho_3 > 1, \rho_{1,2} < 1$

A restriction on ρ_i is needed for this system to behave in a stable manner. Both θ_2 and θ_3 are positive, the restriction that is needed to have all fractions positive is $\rho_3 < 1/(1-\rho_2)$, the proof for this can be found in Appendix D.

The modes graph can be seen in Figure 3.3. Each mode is presented as a circle with the mode labelled in it. A blue arrow is the transition that the system can undergo. As can be seen, a mode can have multiple outgoing (or incoming) arrows. This indicates a decrease in both queues and the system can converge to two modes. Absorbing modes are marked in a different colour, as is the case for mode 1 in the figure below.

The fifth mode is a mode in which the system can never stay, the ODEs for this mode are presented below.

$$Mode\ 5 - \{1,0,0\}$$

$$\dot{Y}_1 = -1 < 0$$

$$\dot{Y}_2 = 0$$

$$\dot{Y}_3 = \rho_3 - 1 > 0$$

The change in Y_3 is positive, while it was empty to start with. This instantly changes the mode from $\{1,0,0\}$ to $\{1,0,1\}$, these instant changes are represented as the light blue arrows in the modes graph. All modes entering mode 5 are redirected to the sixth mode, an example of this is mode 7, shown as a blue arrow.

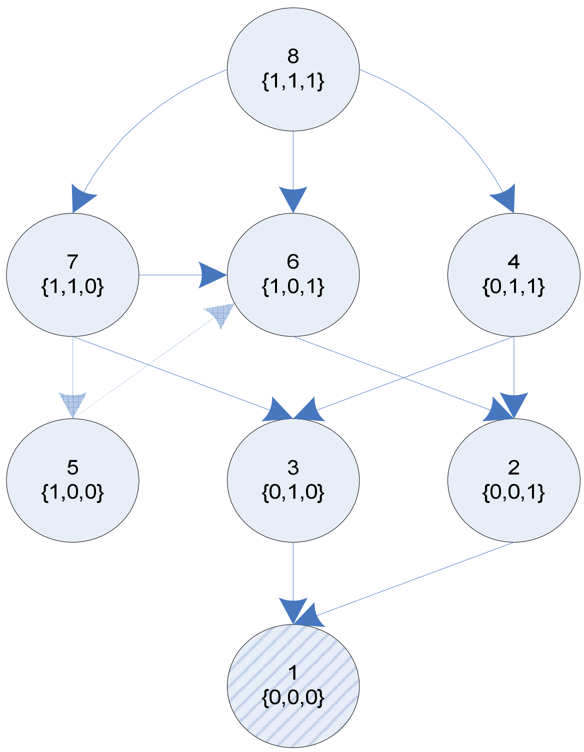

Figure 3.3 – Modes graph for a system with three machines with $\rho_3 > 1, \rho_{1,2} < 1$.

It can be seen that from any mode, the system converges to a mode in which all the queues are empty. All $0 < \theta_i < 1$ and therefore the system is asymptotically stable. Figure 3.4 shows a generated trajectory, it can be seen that from $\underline{X}(0) = \{3,2,1\}$ the system moves to the origin, the mode in which all queues are empty.

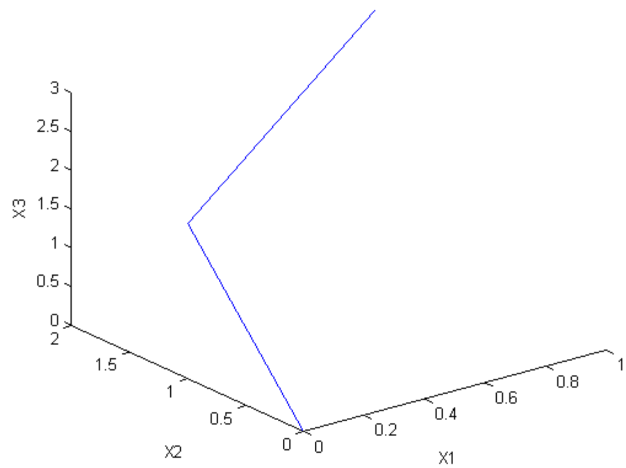

Figure 3.4 – Trajectory for $\rho_3 > 1, \rho_{1,2} < 1$.

Case 3 : $\rho_1 < 1, \rho_{2,3} > 1$

If no restrictions are used on ρ_i, the system can behave in both a stable or an unstable manner. A restriction is needed to get θ_2 positive, this is $\rho_2 < 1/(1-\rho_1)$. With this boundary, all $0 < \theta_i < 1$ and the system behaves in a stable manner in an equilibrium point. The proof for this is shown in Appendix D.

The modes graph is shown in Figure 3.5. There are two modes in which the system cannot stay and will leave instantly. There is only one absorbing state, the first mode.

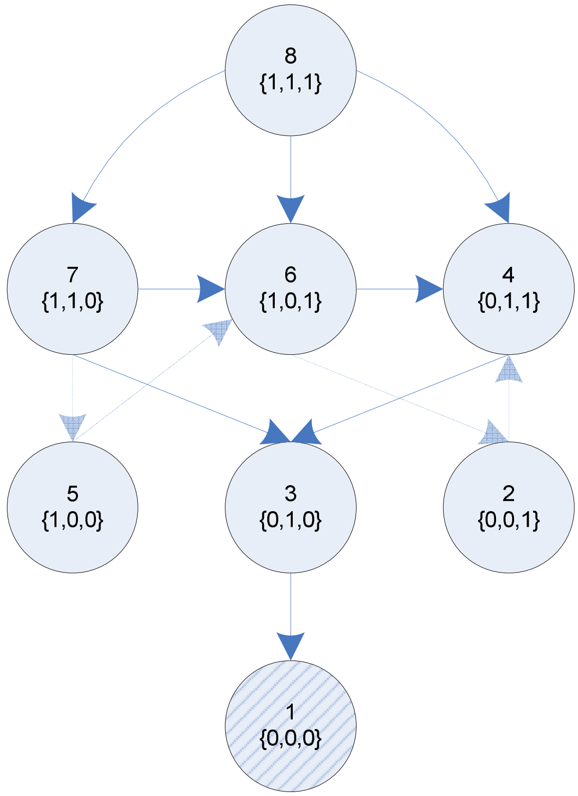

Figure 3.5 – Modes graph for a system with three machines with $\rho_1 < 1, \rho_{2,3} > 1$.

The modes graph shows that from all modes, mode 1 is reached eventually. Also, all $0 < \theta_i < 1$ and this proves the system is asymptotically stable. Figure 3.6 shows a generated trajectory with $\underline{X}(0) = \{3,2,1\}$, the system moves to the origin, showing that the system is indeed asymptotically stable. The trajectory moves from $\{0,2,0\}$ to $\{0,0,0\}$.

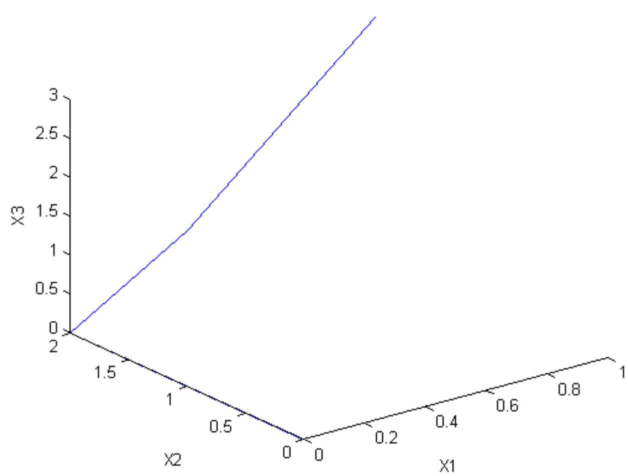

Figure 3.6 – Trajectory for $\rho_1 < 1, \rho_{2,3} > 1$.

Case 4 : $\rho_i > 1$

There are no additional restrictions on ρ_i needed. All θ_i are positive and therefore $0 < \theta_i < 1$. The proof can be found in Appendix D.

Figure 3.7 shows the modes graph, all three modes with one queue non-empty are left instantly and a cyclic behaviour is witnessed between modes 7, 6 and 4. The first mode cannot be reached, however, this does not necessarily mean that the system is not asymptotically stable.

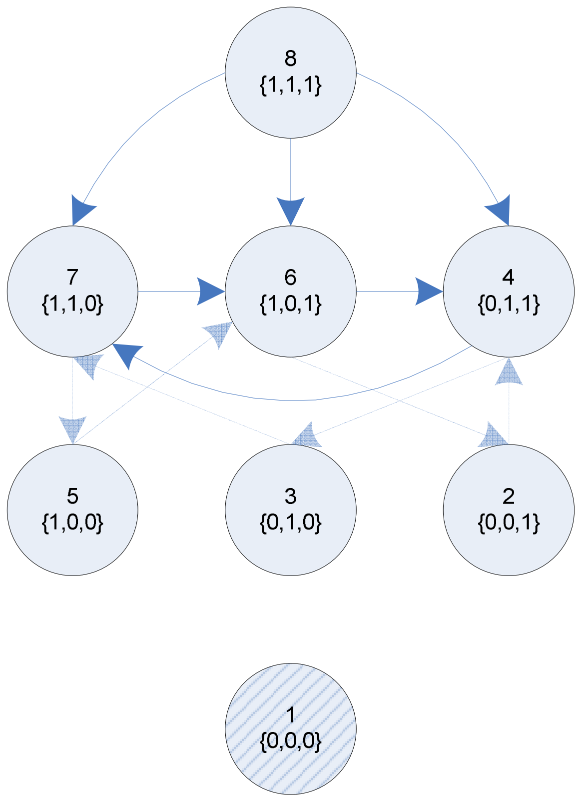

Figure 3.7 – Modes graph for a system with three machines with $\rho_i > 1$.

The system continually changes between the modes 7, 6 and 4. The systems stay in modes 2, 3 and 5 for an infinitely small time and therefore the queues in that mode describe the starting values of the queues in the next mode (either 7, 6 or 4). To be able to test if this system is asymptotically stable, we assume that the system is in mode 3 for the k-th time. The queues can be described by the variable $\underline{Y}_3^k$. Upon switching to the next mode 5, the system can be described by $\underline{Y}_5^k$. $\underline{Y}_2^k$ describes the queues in mode 2.

$$\underline{Y}_3^k = \begin{pmatrix} 0 \\ Y_2 \\ 0 \end{pmatrix},\ \underline{Y}_5^k = \begin{pmatrix} Y_2(\rho_1 - 1) \\ 0 \\ 0 \end{pmatrix},\ \underline{Y}_2^k = \begin{pmatrix} 0 \\ 0 \\ Y_2(\rho_1 - 1)(\rho_3 - 1) \end{pmatrix}$$

When finally returning to the mode in which we started, 3, the queues are described by $\underline{Y}_3^{k+1}$.

$$\underline{Y}_3^{k+1} = \begin{pmatrix} 0 \\ Y_2(\rho_1 - 1)(\rho_2 - 1)(\rho_3 - 1) \\ 0 \end{pmatrix}$$

It can now be seen that

$$\underline{Y}_3^{k+1} = (\rho_1 - 1)(\rho_2 - 1)(\rho_3 - 1)\underline{Y}_3^k \tag{3.3}$$

So, the system is asymptotically stable when $\prod_{i=1}^{3}(\rho_i - 1) < 1$. It is asymptotically unstable when $\prod_{i=1}^{3}(\rho_i - 1) > 1$ and marginally stable for $\prod_{i=1}^{3}(\rho_i - 1) = 1$. Figure 3.8 clearly shows the difference, both started with the same initial condition, close to $\underline{X}(0) = \{2,1,2\}$. The figure on the right increases over time, whereas the figure on the left continually decreases.

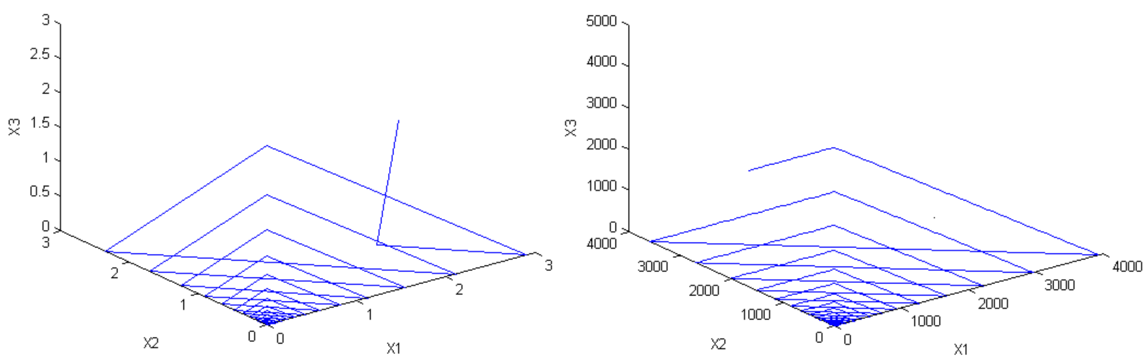

Figure 3.8 – Trajectories for $\rho_i > 1$, for the left figure $\prod_{i=1}^{3}(\rho_i - 1) < 1$, and for the right $\prod_{i=1}^{3}(\rho_i - 1) > 1$

In this section it became clear that for some cases, restrictions are needed so that all $0 < \theta_i < 1$. In the last case, in which $\rho_i > 1$ the first mode cannot be reached and the system stays in a permanent cyclic behaviour between three modes. This cycle is asymptotically stable if a certain condition is met. In the next section the (asymptotical) stability of a system with four machines is analyzed.

3.3 Analysis of a system with four machines

A similar analysis as presented in section 3.1 and 3.2 is used to analyze a cyclic push-pull system with four machines.

First, using the steady state flow equations the expressions for θ_i are derived and a number of cases are presented with different combinations of ρ_i.

In this section a new phenomenon, a transient cyclic behaviour is encountered. This is different from the case with all $\rho_i > 1$ for $M = 3$ presented in section 3.2, the system can leave this cycle. Also, there is a case in which some parts of the state space are asymptotically stable and the rest is not. Both cases are analyzed in more detail.

Using the steady state flow equations, three equations can be derived by using the conservation of mass, solving these equations obtains the expressions for θ_i.

$$
\begin{aligned}
\theta_1 &= \frac{\rho_2\rho_3\rho_4 - \rho_3\rho_4 + \rho_4 - 1}{\rho_1\rho_2\rho_3\rho_4 - 1} = \frac{\rho_4(1-(1-\rho_2)\rho_3)-1}{\rho_1\rho_2\rho_3\rho_4 - 1} \\
\theta_2 &= \frac{\rho_1\rho_3\rho_4 - \rho_1\rho_4 + \rho_1 - 1}{\rho_1\rho_2\rho_3\rho_4 - 1} = \frac{\rho_1(1-(1-\rho_3)\rho_4)-1}{\rho_1\rho_2\rho_3\rho_4 - 1} \\
\theta_3 &= \frac{\rho_1\rho_2\rho_4 - \rho_1\rho_2 + \rho_2 - 1}{\rho_1\rho_2\rho_3\rho_4 - 1} = \frac{\rho_2(1-(1-\rho_4)\rho_1)-1}{\rho_1\rho_2\rho_3\rho_4 - 1} \\
\theta_4 &= \frac{\rho_1\rho_2\rho_3 - \rho_2\rho_3 + \rho_3 - 1}{\rho_1\rho_2\rho_3\rho_4 - 1} = \frac{\rho_3(1-(1-\rho_1)\rho_2)-1}{\rho_1\rho_2\rho_3\rho_4 - 1}
\end{aligned}
\tag{3.4}
$$

It can be seen that the denominator can be either positive or negative, depending on the product $\rho_1\rho_2\rho_3\rho_4$.

Now, the stability for all the combinations of ρ_i is analyzed using the expressions derived earlier in (3.4).

The ODEs for $M = 4$ depend on the push and pull parameters μ_i and λ_i, or $\rho_i = \lambda_i / \mu_i$. There are $2^4 = 16$ different cases, each with $2^4 = 16$ modes. First, the cases will be looked at and afterwards the ODEs for each case are given. For these cases, it is investigated if the system is stable. If they are stable the asymptotically stable is examined. For every case a modes graph is made, instant transitions are not marked anymore to increase readability.

The variable ρ_i is always positive and can be either $\rho_i < 1$ or $\rho_i > 1$, the singular $\rho_i = 1$ is not taken into account. This gives sixteen cases in total. Some of these modes are the same, i.e. only the indices are different, and therefore are not examined. Without loss of generality we can distinguish six cases.

1. $\rho_i < 1$
2. $\rho_1 > 1, \rho_{2,3,4} < 1$
3. $\rho_{1,2} > 1, \rho_{3,4} < 1$
4. $\rho_{1,3} > 1, \rho_{2,4} < 1$
5. $\rho_{1,2,3} > 1, \rho_4 < 1$
6. $\rho_i > 1$

Once again, case 1 in which $\rho_i < 1$ is always stable and also asymptotically stable.

Case 4 is unstable as the numerator of θ_1 is always negative and the numerator of θ_2 is always positive. The proof for this can be found in Appendix E. Because of this, the signs of θ_1 and θ_2 are opposite and the solution is not feasible, the system is unstable. Therefore, the fourth case is not analyzed as well.

The modes are labelled with binary digits, these state whether a queue is full or not, $\{X_1, X_2, X_3, X_4\}$. The ODEs for the cases can be found in Appendix C.

Case 2 : $\rho_1 > 1, \rho_{2,3,4} < 1$

A number of restrictions on ρ_i are needed for this system to behave in a stable manner. The numerator of θ_1 is negative and therefore the denominator has to be negative, which yields $\rho_1\rho_2\rho_3\rho_4 < 1$. Because of this, all numerators of θ_i have to be negative. This gives $\rho_1 < 1/(1-\rho_4(1-\rho_3))$ for the numerator of θ_2 and $\rho_1 < (1-\rho_3(1-\rho_2))/(\rho_2\rho_3)$ for the numerator of θ_4. The expression $1-\rho_1(1-\rho_4)$ for θ_3 has to be positive or else it will conflict with the other restrictions. The proof for these statements can be found in Appendix E.

Figure 3.9 shows the modes graph, there is only one absorbing state, the first mode. This makes this case asymptotically stable.

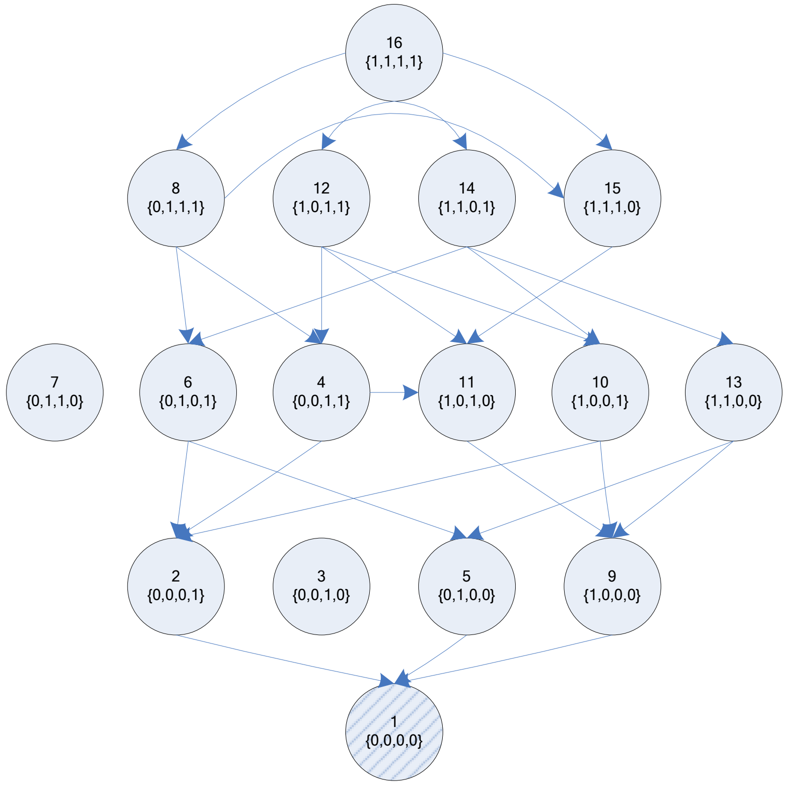

Figure 3.9 – Modes graph for a system with four machines with $\rho_1 > 1, \rho_{2,3,4} < 1$.

Case 3 : $\rho_{1,2} > 1, \rho_{3,4} < 1$

In this case, the numerators of all θ_i can be either positive or negative. This implies that there are multiple cases, one in which $\rho_1\rho_2\rho_3\rho_4 < 1$ and one in which $\rho_1\rho_2\rho_3\rho_4 > 1$.

Case 3a : $\rho_{1,2} > 1, \rho_{3,4} < 1$ and $\rho_1\rho_2\rho_3\rho_4 < 1$

The denominators of all θ_i are negative, therefore all numerators have to be negative as well. The expression $1-(1-\rho_4)\rho_1 > 0$ has to hold, otherwise θ_3 cannot be positive. The proof for these statements can be found in Appendix E.

Figure 3.10 shows the modes graph, there is only one absorbing state, the first mode. This makes this case asymptotically stable.

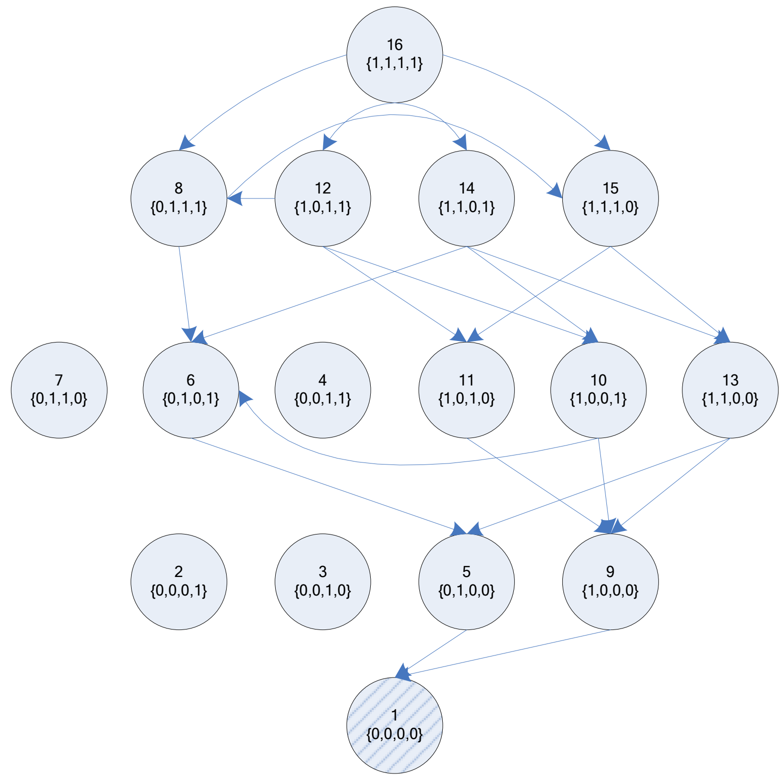

Figure 3.10 – Modes graph for a system with four machines with $\rho_{1,2} > 1, \rho_{3,4} < 1$ and $\rho_1\rho_2\rho_3\rho_4 < 1$.

Case 3b : $\rho_{1,2} > 1, \rho_{3,4} < 1$ and $\rho_1\rho_2\rho_3\rho_4 > 1$

The denominators of all θ_i are positive, therefore all numerators have to be positive as well. The expression $1-(1-\rho_4)\rho_1 > 0$ has to hold, otherwise it conflicts with the other requirements. The proof for these statements can be found in Appendix E.

Figure 3.11 shows the modes graph, there are two absorbing state, the first and ninth mode. The first mode can be reached, this implies that for some parts of the state space the system is asymptotically stable and for the rest it is only stable. This behaviour is analyzed in subsection 3.3.1.

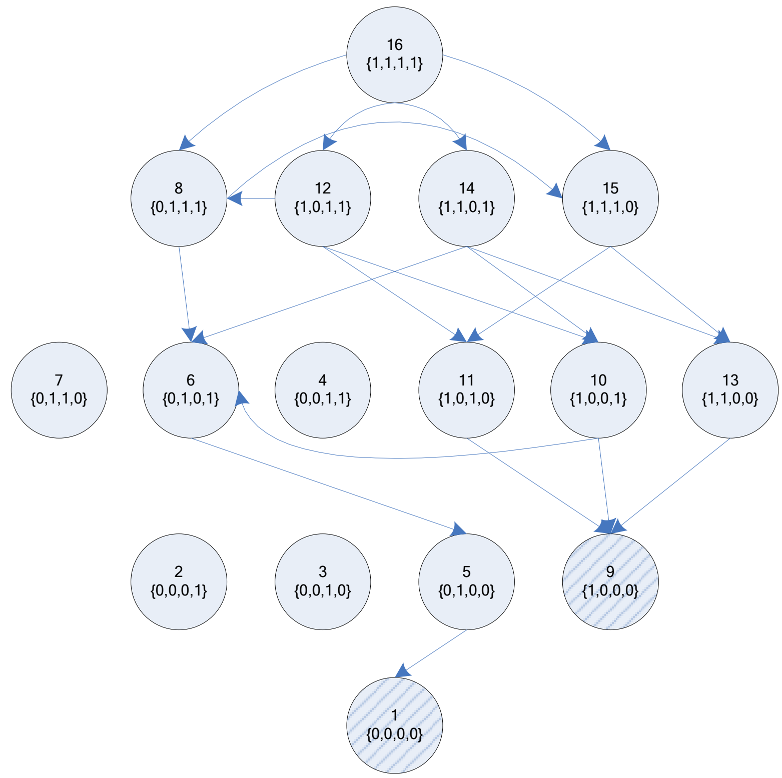

Figure 3.11 – Modes graph for a system with four machines with $\rho_{1,2} > 1, \rho_{3,4} < 1$ and $\rho_1\rho_2\rho_3\rho_4 > 1$.

Case 5 : $\rho_{1,2,3} > 1, \rho_4 < 1$

The numerator of θ_2 and θ_4 is always positive, the other numerators can be either positive or negative. Therefore, only $\rho_1\rho_2\rho_3\rho_4 > 1$ is valid. Thus, the numerators of θ_1 and θ_3 have to be positive. The proof for these statements can be found in Appendix E.

Figure 3.12 shows the modes graph, there are two absorbing state, the first and fifth mode. The first mode cannot be reached, therefore this case is stable but not asymptotically stable.

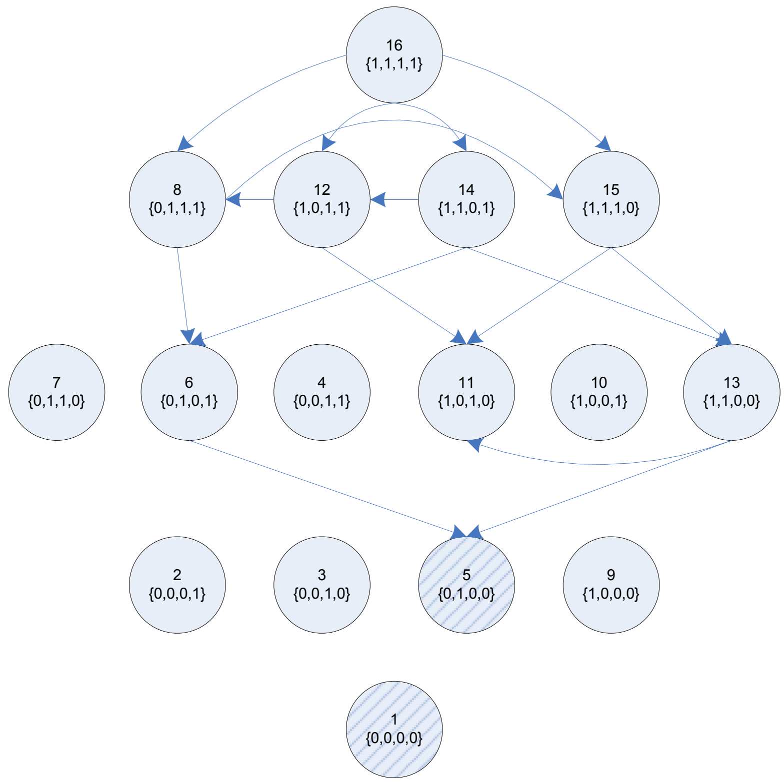

Figure 3.12 – Modes graph for a system with four machines with $\rho_{1,2,3} > 1, \rho_4 < 1$.

Case 6 : $\rho_i > 1$

There are no additional requirements needed as all numerators and denominators are positive and therefore all $\theta_i > 0$.

Figure 3.13 shows the modes graph, there are three absorbing states, the first, sixth and eleventh mode. A transient cyclic behaviour is witnessed for modes 15, 14, 12 and 8. In subsection 3.3.2 it is analyzed if the system can stay in this cycle for infinite time.

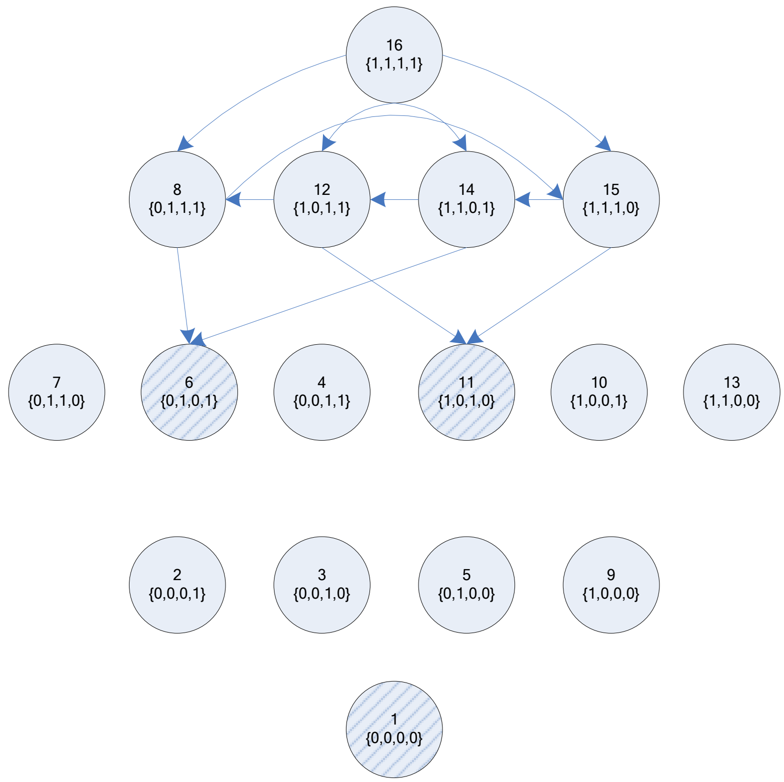

Figure 3.13 – Modes graph for a system with four machines with $\rho_i > 1$.

3.3.1 Partially asymptotically stable system

The behaviour witnessed in case 3b from section 3.3 is analyzed in greater detail. We examine what part of the state space is asymptotically stable and what part is just stable. The modes graph is printed below, the ODEs can be found in Appendix C.

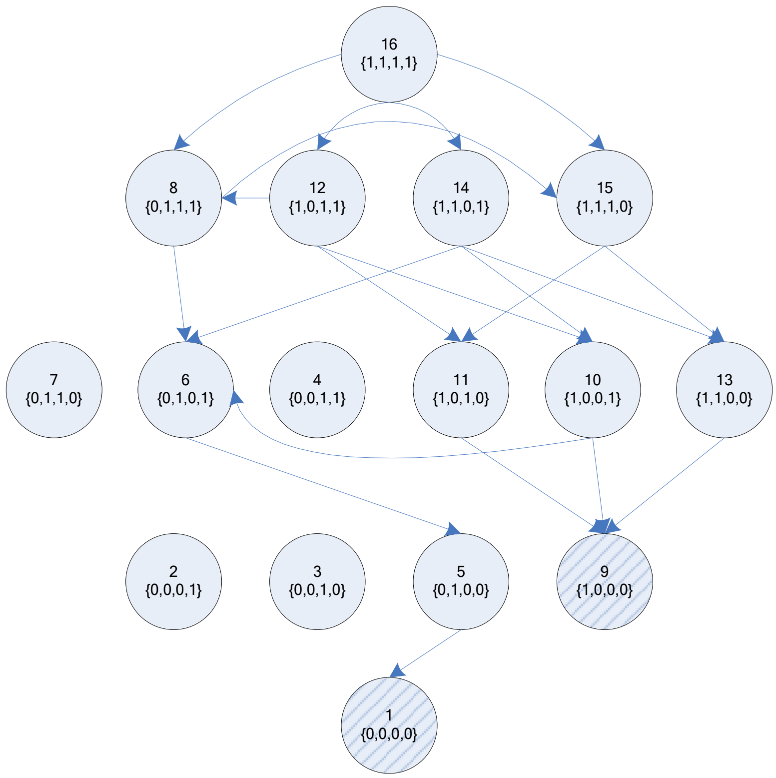

Figure 3.14 - Modes graph for a system with four machines with $\rho_{1,2} > 1, \rho_{3,4} < 1$ and $\rho_1\rho_2\rho_3\rho_4 > 1$.

As can be seen in Figure 3.14, there is a fixed route in which the system converges to the absorbing mode in the first mode $\{0,0,0,0\}$, this route is $6 \rightarrow 5 \rightarrow 1$. Any mode that reaches one of these modes will always converge to mode 1 and is therefore asymptotically stable.

There are five possible routes that lead to one of the three modes in the route to mode 1. These routes are

1. $16 \rightarrow 8 \rightarrow 6$
2. $16 \rightarrow 12 \rightarrow 8 \rightarrow 6$
3. $16 \rightarrow 12 \rightarrow 10 \rightarrow 6$
4. $16 \rightarrow 14 \rightarrow 10 \rightarrow 6$
5. $16 \rightarrow 14 \rightarrow 6$

The five routes are analyzed by starting from mode 16 and working down to mode 6. Y_i^j means the time needed to empty queue i in mode j. The section covers the results of the analysis, the proof can be found in Appendix F.

Route 1 : $16 \to 8 \to 6$

For this route there are two transitions that take place, both transitions have restrictions on the variables, combining these two restrictions gives us the total restriction

$$\left\{(Y_1^{16}, Y_2^{16}, Y_3^{16}, Y_4^{16}) \middle| Y_1^{16} < Y_2^{16} \wedge Y_1^{16} < Y_3^{16} < Y_4^{16}\right\} \tag{3.5}$$

This can hold and at represents a 4D volume in the state space.

Route 2 : $16 \to 12 \to 8 \to 6$

For this route to occur, the system must undergo three transitions, which leads to three different restrictions. These restrictions are derived in Appendix F and the total restriction is presented below.

$$\left\{(Y_1^{16}, Y_2^{16}, Y_3^{16}, Y_4^{16}) \middle| Y_2^{16} < Y_1^{16} < Y_3^{16} + \rho_3(Y_1^{16} - Y_2^{16}) < Y_4^{16}\right\} \tag{3.6}$$

It can be noted that in stead of two different restrictions as could be seen in route 1, there is only one restriction. This is always the case for a route with three transitions. This restriction can hold as well and spans a 4D volume in the state space.

Route 3 : $16 \to 12 \to 10 \to 6$

Once again, a route with three transitions is analyzed. The total restriction holds and spans a 4D volume.

$$\left\{(Y_1^{16}, Y_2^{16}, Y_3^{16}, Y_4^{16}) \middle| Y_2^{16} < Y_3^{16} < Y_1^{16} < \frac{1}{1-(1-\rho_3)\rho_4}\left[Y_4^{16} + \rho_4 Y_3^{16} + \rho_3\rho_4 Y_2^{16}\right]\right\} \tag{3.7}$$

Route 4 : $16 \to 14 \to 10 \to 6$

In the fourth route the system has to undergo three transitions to be able to go to mode 6 and by that to the first mode. The total restriction holds because we can choose Y_4^{16} as large as we want and by that fulfilling the restriction.

$$\left\{(Y_1^{16}, Y_2^{16}, Y_3^{16}, Y_4^{16}) \middle| Y_3^{16} < Y_2^{16} < Y_1^{16} < \frac{Y_4^{16} - \rho_3\rho_4 Y_2^{16} + (2\rho_3 - 1)\rho_4)Y_3^{16}}{1-(1-\rho_3)\rho_4}\right\} \tag{3.8}$$

Route 5 : $16 \rightarrow 14 \rightarrow 6$

In the last mode, only two transitions take place. As was said earlier, for a route with two transitions there are two restrictions as well. The restriction spans a 4D volume.

$$\left\{ (Y_1^{16}, Y_2^{16}, Y_3^{16}, Y_4^{16}) \middle| Y_3^{16} < Y_1^{16} < Y_2^{16} \wedge Y_1^{16} < \frac{1}{1-\rho_4}(Y_4^{16} - \rho_4 Y_3^{16}) \right\} \tag{3.9}$$

For all five routes the restriction spans a 4D volume, this indicates that the system can indeed be asymptotically stable if one of the five restrictions is met. In the next subsection a closer look on a transient cyclic behaviour is given.

3.3.2 Transient cyclic behaviour

The behaviour witnessed in case 6 from section 3.3 is analyzed in greater detail. We examine if it is possible to stay in this cycle instead of escaping to one of the two absorbing states. The modes graph is printed below, the ODEs can be found in Appendix C.

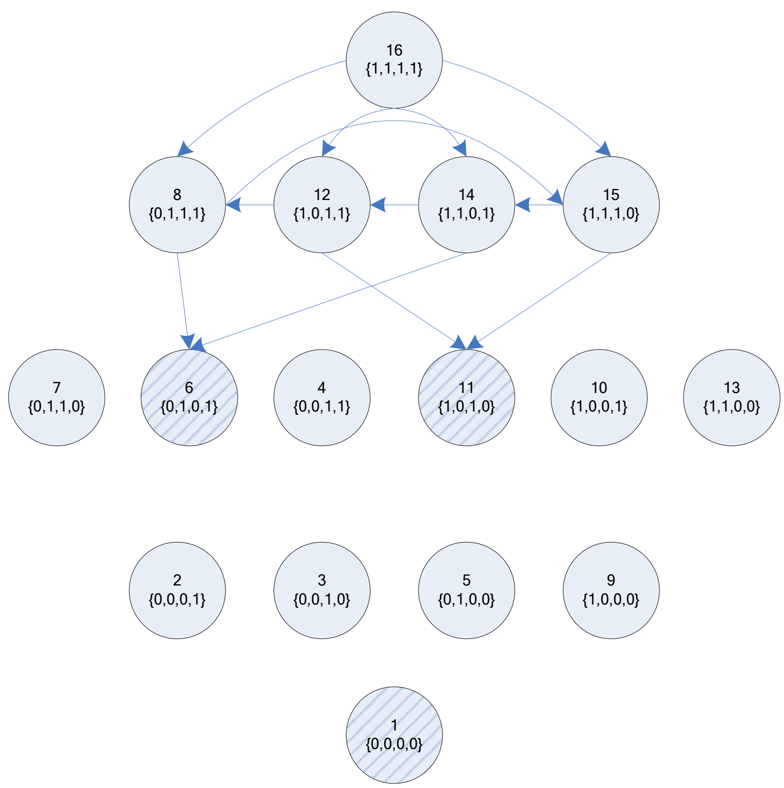

Figure 3.15 – Modes graph for a system with four machines with $\rho_i > 1$.

As can be seen in Figure 3.15, there is a cyclic behaviour between the modes $15 \rightarrow 14 \rightarrow 12 \rightarrow 8 \rightarrow 15 \rightarrow \ldots$. It is analyzed if there is a certain parameter setting that allows the system to stay in this cycle for infinite time without the queues increasing continually. If there are parameters that fulfil the restrictions the system is asymptotically stable. The analysis starts in the 15-th mode which we enter for the k-th time. In this section, only the results are presented, the proof can be found in Appendix G. To be able to stay within the cycle the following condition has to be met

$$\left[\frac{\rho_1 - 1}{\rho_4} + 1\right] Y_3^{15} < Y_2^{15} < \left[\rho_1 - \frac{(\rho_1 - 1)(\rho_4 - 1)}{\rho_3 + \rho_4 - 1}\right] Y_3^{15} \tag{3.10}$$

If parameters can be found so that the system stays within the cycle, it is not necessarily asymptotically stable. For it to be asymptotically stable, the queues have to go to zero for infinite time. In Appendix G, a set of equations is derived that describe the queues after one cycle, starting and returning in mode 15. This can be written in the form $\underline{Y}(k+1) = A(\rho)\underline{Y}(k)$ as

$$\begin{pmatrix} Y_2^{15}(k+1) \\ Y_3^{15}(k+1) \end{pmatrix} = \begin{pmatrix} \rho_4(\rho_2 - 1) & -(\rho_2 - 1)(\rho_1 + \rho_4 - 1) \\ -(\rho_4 + \rho_3 - 1) & \rho_3\rho_1 + \rho_4 - 1 \end{pmatrix} \begin{pmatrix} Y_2^{15}(k) \\ Y_3^{15}(k) \end{pmatrix} \tag{3.11}$$

For the system to be asymptotically stable, the eigenvalues of matrix $A(\rho)$ have to be in the (complex) unit disc. If $|\lambda_i| > 1$, for any *i*, a queue will grow. Using the Jury test and the characteristic polynomial of the matrix, it can be found if $|\lambda_i| < 1$ for all *i*. The characteristic polynomial of $A(\rho)$ can be expressed as

$$\lambda^2 + a_1\lambda + a_2 = 0 \tag{3.12}$$

All roots of (3.12) are inside the unit disc if

$$\begin{aligned} a_2 &< 1 \\ a_2 &> -1 + a_1 \\ a_2 &> -1 - a_1 \end{aligned} \tag{3.13}$$

Rewriting this with the use (3.11) gives the following condition

$$-2 + \rho_1\rho_3 + \rho_2\rho_4 < (\rho_1 - 1)(\rho_2 - 1)(\rho_3 - 1)(\rho_4 - 1) < 1 \tag{3.14}$$

For the system to be asymptotically stable, both (3.10) and (3.14) need to hold. The latter of the two cannot hold and therefore the system is not asymptotically stable for any parameter setting. If it can stay in the cycle (if (3.10) is met) the queues will increase over time.

In this chapter a complete characterization of systems with two, three or four machines are given. For two machines, only one region is asymptotically stable. A system with three machines is asymptotically stable for a number of regions. A permanent cyclic behaviour is witnessed if $\rho_i > 1$, after analyzing this behaviour it is concluded that a certain condition has to be met so that the system is asymptotically stable. A system with four machines is analyzed by looking at six cases. For two of these cases an in-depth analysis of asymptotical stability is done. It is concluded that the system always leaves the transient cycle and that a system can be partially asymptotically stable.

In the next chapter an open question regarding the cyclic behaviour seen for $M = 3$ is analyzed by the means of simulation.

Chapter 4 Attacking open questions by simulation

In this chapter a closer look is given at a recurring behaviour and its asymptotic stability for arbitrary M. This behaviour is the permanent cyclic behaviour first witnessed for $M = 3$ and all $\rho_i > 1$, see section 3.2, case 4. For $M = 4$ this behaviour becomes transient in stead of permanent. It is believed that for all odd number of machines and all $\rho_i > 1$, the system moves through a permanent cycle.

In the analysis, all ρ_i are equal and labelled $\bar{\rho}$. By means of simulation, the critical value for which the system becomes marginally stable is sought. The variable $\bar{\rho}_K$ is this critical value, for which $\bar{\rho}_K$ $\forall\ 1 < \bar{\rho} < \bar{\rho}_K$, $M = 2K + 1$, $K = 1, 2, 3, \ldots$ and $\rho_i = \bar{\rho}$ the system is asymptotically stable.

The simulation is done with the help of the M-files presented in Appendix A, for $K = 1, 2, 3, 4, 5, 6, 20$. The outcome of the M-file are trajectories for all X_i, it can be seen if the system asymptotically stable or not. The queues either decrease of increase over time. By picking values for $\bar{\rho}$ which are asymptotically stable or only stable we can converge to the critical value $\bar{\rho}_K$. The results are plotted in Figure 4.1.

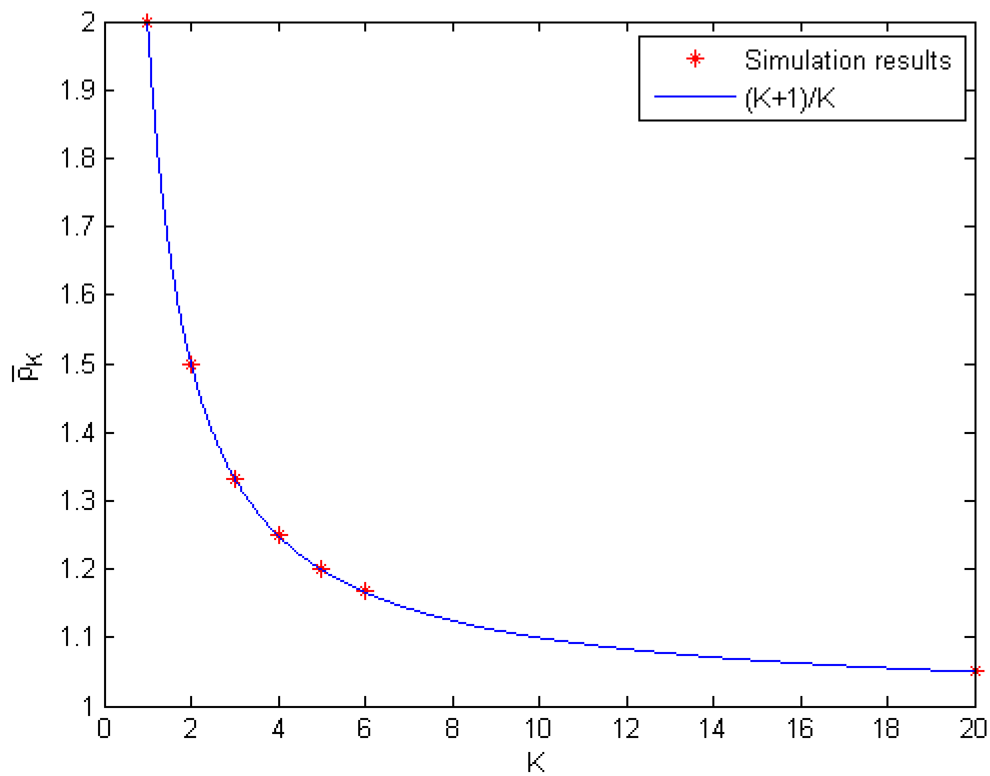

Figure 4.1 – Simulation results and fitted curve.

It can be seen that the fitted curve $(K+1)/K$ corresponds with the simulation results. All systems with $M = 2K+1$ and $(K+1)/K > \overline{\rho} > 1$ behave asymptotically stable.

Chapter 5 Conclusion

With the increasing complexity and costs of manufacturing systems, it is important to study and analyze these production systems so that an insight is gained into the behaviour of the network of machines.

In this paper, an analysis of a cyclic push-pull system has been done. The analysis focuses on the influence of a pull-first policy on (asymptotical) stability, for both general and specific cases. Such an analysis is useful as the cyclic push-pull system allows production of an arbitrary number of products, it allows full utilization of all machines and it is often stable.

An expression for the fraction of time spent on processing, θ, governs the stability. This variable depends on the push- and pull-parameters, or ρ_i, If all θ_i are within the range $[0,1]$ the system will behave in a stable manner once it reaches an equilibrium point. A system is asymptotically stable if it is stable and all queues go to zero for $t \to \infty$.

A complete characterization for systems with two, three of four machines has been made. For systems with an arbitrary number of machines and $\rho_i < 1$ the system is always asymptotically stable.

A system with an odd number of machines and $\rho_i > 1$ will transition through a permanent cycle, for such cyclic behaviour the asymptotical stability depends on ρ_i. Through simulation a conjecture was made which states that, if all ρ_i are equal and labelled $\bar{\rho}$, all systems consisting of $M = 2K+1,\ K = 1,2,...$ with $\bar{\rho}$ for which $(K+1)/K > \bar{\rho} > 1$ are asymptotically stable.

Cyclic push-pull systems with an even number of machines and $\rho_i > 1$ have a transient cyclic behaviour. For a system with four machines there are two conditions that need to be met so that it is asymptotically stable, one of these restrictions cannot hold and therefore it is only stable.

In this paper, most of the symmetric cases are analyzed and general conditions are given for which these systems are asymptotically stable. In future research, one could try to find a general characterization for asymptotical stability for asymmetric cases (a combination of $\rho_i < 1$ and $\rho_i > 1$) and for systems with an even number of machines and $\rho_i > 1$.

References

[1] Y. Nazarathy and G. Weiss. Positive Harris recurrence and diffusion scale analysis of a push-pull queueing network. Performance Evaluation – To appear, 2008.
[2] A. Kopzon, Y. Nazarathy and G. Weiss. A Push-Pull network with infinite supply of work. Springer Science+Business Media, LLC, 2009.
[3] Khalil, Hassan K. Nonlinear systems (pg. 134-135). Prentice-Hall, Inc., Upper Saddle River, New Jersey, 1996.
[4] J.A.W.M. v. Eekelen. Modelling and control of discrete event manufacturing flow Lines. Universiteitsdrukkerij Technische Universiteit Eindhoven, 2007.

Appendix A Matlab files

In this appendix the Matlab files that are used for this project are presented.

A.1 Symbolic expression for theta

This M-file derives the symbolic expressions for θ_i for all M. It works by creating M equations of $\rho_i\theta_i = (1-\theta_{i+1})$, and solving them with a left matrix divide. If $i = M$, $i+1=1$, this is the variable n in the Matlab code.

```
% TU/e Mechanical Engineering, Systems Engineering
% Bachelor project: Dynamics of an Abstract Production System
%
% Student   Jori Selen
% Advisors  Erjen Lefeber, Yoni Nazarathy
%
% Date created 14 march 2010
% ----------------------------------------------------------------
% Deriving symbolic expressions for theta for all M

function theta_sym = derive_theta_symbolic(M)

A = sym(zeros(M));
b = ones(M,1);

for i = 1:M
    if i == M
        n = 1;
    else
        n = i + 1;
    end

    A(i,i) = ['rho',num2str(i)];
    A(i,n) = 1;

end

theta_sym = A\b;
```

A.2 Generating trajectories for X(t)

A combination of four M-files derive the trajectories for the queues, the first file is the main file in which all the other functions are called. It also checks for stability and if the system is in an absorbing state. In the other three files the ODEs per mode are derived, the values for theta are computed and the trajectories are plotted.

A.2.1 Trajectories.m

```
% TU/e Mechanical Engineering, Systems Engineering
```

```
% Bachelor project: Dynamics of an Abstract Production System
%
% Student   Jori Selen
% Advisors  Erjen Lefeber, Yoni Nazarathy
%
% Date created 14 march 2010
% ---------------------------------------------------------------
% Generating trajectories for X(t), T_1, T_2 and D

clc, clear all, format short

%% Input variables

% Number of machines in the system
M = 2;

% Push and pull-parameters
rho = [0.1; 0.1];
mu  = ones(M,1);
lam = rho.*mu;

% Starting values for the queues
X0 = [2; 1];

%% Checking if the input is correct

input = ( M == length(find(rho > 0)) && M == length(find(mu > 0)) && M
== length(find(X0 > 0)) );

if ~input
    error('The input is incorrect');
end

%% Checking for asymptotical stability
% Continuing the analysis if theta_i > 0.

theta = derive_theta(M,rho);

unstable = find(theta < 0);

if unstable
    error(['theta(',num2str(unstable),') is smaller than 0, the system
is unstable, ending the M-file']);
end

%% Generating the trajectories
% The loop stops if the system reaches mode 1 {0,0,...,0}
% or a maximum number of iterations are performed.

iter_max = 500;

X = X0;
mode = ( X > 0 );
```

```
% Initial values, the queues start at X0, time at 0, the amount of work
on
% activity 1 or 2 (T_1 and T_2) at 0, fluid output (D) at 0 and the
% iteration count at 2.
X_tr(:,1)   = X;
t(1)        = 0;
T_1(1:M,1) = 0;
T_2(1:M,1) = 0;
D(1:M,1)    = 0;

iter = 2;

while ~(iter == iter_max)

    [ODEs fcs] = derive_ODEs(M,mode,rho,mu,lam);

    % mode_ODEs gives the queues in which there is a change (binary)
    % If mode(i) = 1 and ODEs(i) = 0 for the same i, mode_ODEs(i) = 1.
    mode_ODEs = mode + abs(ODEs);
    mode_ODEs(mode_ODEs > 1) = 1;

    % If the mode is a sink, the while loop is exitted
    sink_check = ODEs;
    sink_check(sink_check > 1) = 1;
    sink = ( isequal(mode,zeros(M,1)) || isequal(mode,sink_check) );

    if sink
        display(['Sink has been reaced in mode {',num2str(mode'),'}']);
        display(['ODEs: ',num2str(ODEs')])
        break
    end

    % The if statement checks if an empty buffer is increasing,
    % if this is the case, it changes mode.
    if isequal(mode,mode_ODEs)

        % Resetting the time needed to empty a queue
        t_empty = -1e2;

        % Finding the minimum time required to empty a queue
        for i = 1:M
            t_empty_X(i) = X(i)/ODEs(i);
        end

        for i = find(t_empty_X < 0)
            t_empty = max(t_empty,t_empty_X(i));
        end

        t_empty = abs(t_empty);

        % Updating the time vector
        t(iter) = t(iter-1) + t_empty;

        % Updating the queues
```

```
        for i = 1:M
            X(i) = X(i)+ODEs(i)*t_empty;
        end

        % Saving the values for X in an array
        X_tr(:,iter) = X;

        % Updating the mode
        mode = (X > 0);

        % The amount of work on either pushing (T_1) or pulling (T_2)
and
        % the amount of fluid output (D) are updated.
        for i = 1:M
            T_1(i,iter) = T_1(i,iter-1) + fcs(i,1)*t_empty;
            T_2(i,iter) = T_2(i,iter-1) + abs(fcs(i,2))*t_empty;
            D(i,iter) = D(i,iter-1) + abs(fcs(i,2))*mu(i)*t_empty;
        end

        iter = iter + 1;
    else
        mode = mode_ODEs;
    end

end

%% Plotting the trajectories

plot_tr(M,X_tr,T_1,T_2,D,t)
```

A.2.2 derive_ODEs.m

```
% TU/e Mechanical Engineering, Systems Engineering
% Bachelor project: Dynamics of an Abstract Production System
%
% Student   Jori Selen
% Advisors  Erjen Lefeber, Yoni Nazarathy
%
% Date created 21 march 2010
% ---------------------------------------------------------------
% Deriving the ODEs for all M, rho and mode
%
% fcs is a variable that contains all factors in front of lambda
% and mu respectively. It is used to derive T_i,j and D_i.
%  If Xdot = (1-rho(3))*lambda - (1-rho(3))*rho(4)*mu, fcs will
% be [1-rho(3) (1-rho(3))*rho(4)].


function [ ODEs fcs ] = derive_ODEs(M,mode,rho,mu,lam)

ODEs = zeros(M,1);
fcs = zeros(M,2);

for i = 1:M
```

```
    if i == M
        n = 1;
    else
        n = i + 1;
    end
    if ( mode(i) && mode(n) )
        ODEs(n) = -mu(n); fcs(n,:) = [0 -1];
        rho_l = 1;
        break
    elseif ( mode(i) && ~mode(n) )
        ODEs(n) = 0; fcs(n,:) = [0 0];
        rho_l = 0;
        break
    end
end

if isequal(mode,zeros(M,1))
    rho_l = 1;
end

for i = 1:(M-1)

    l = n;

    if n == M
        n = 1;
    else
        n = l + 1;
    end
    if mode(l)
        if mode(n)
            ODEs(n) = -mu(n); fcs(n,:) = [0 -1];
            rho_l = 1;
        elseif ~mode(n)
            ODEs(n) = 0;fcs(n,:) = [0 0];
            rho_l = 0;
        end
    elseif ~mode(l)
        if mode(n)
            ODEs(n) = (1-rho_l)*lam(n)-mu(n); fcs(n,:) = [1-rho_l -1];
            rho_l = 1;
        elseif ~mode(n)
            if ( ((1-rho_l)*lam(n)) < mu(n) )
                ODEs(n) = 0; fcs(n,:) = [1-rho_l -(1-rho_l)*rho(n)];
                rho_l = (1-rho_l)*rho(n);
            elseif ( ((1-rho_l)*lam(n)) >= mu(n) )
                ODEs(n) = (1-rho_l)*lam(n)-mu(n); fcs(n,:) = [1-rho_l -
1];
                rho_l = 1;
            end
        end
    end
end
```

A.2.3 derive_theta.m

```
% TU/e Mechanical Engineering, Systems Engineering
% Bachelor project: Dynamics of an Abstract Production System
%
% Student   Jori Selen
% Advisors  Erjen Lefeber, Yoni Nazarathy
%
% Date created 14 march 2010
% ---------------------------------------------------------------
% Deriving expressions for theta for all M
% NB These expressions are only valid when the system is stable

function theta = derive_theta(M,rho)

A = zeros(M);
b = ones(M,1);

for i = 1:M
    if i == M
        n = 1;
    else
        n = i + 1;
    end

    A(i,i) = rho(i);
    A(i,n) = 1;

end

theta = A\b;
```

A.2.4 plot_tr.m

```
% TU/e Mechanical Engineering, Systems Engineering
% Bachelor project: Dynamics of an Abstract Production System
%
% Student   Jori Selen
% Advisors  Erjen Lefeber, Yoni Nazarathy
%
% Date created 25 march 2010
% ---------------------------------------------------------------
% Plotting trajectories for X(t), T_1, T_2 and D

function plot_tr(M,X_tr,T_1,T_2,D,t)

if M == 2
    figure, plot(X_tr(1,:),X_tr(2,:))
    xlabel('X1'), ylabel('X2')
elseif M == 3
    figure, plot3(X_tr(1,:),X_tr(2,:),X_tr(3,:))
    xlabel('X1'), ylabel('X2'), zlabel('X3')
else
    figure, hold on
```

```
    for i = 1:M
      subplot(M,1,i), plot(t,X_tr(i,:))
      xlabel('time'), ylabel(['X',num2str(i)])
    end
end

colour = ['b' 'g' 'r' 'c' 'm' 'y' 'k'];
figure

tags = '';

for i = 1:M
    H1 = subplot(3,1,1); plot(t,T_1(i,:),colour(i)), hold on
    xlabel('time'), ylabel('push')
    H2 = subplot(3,1,2); plot(t,T_2(i,:),colour(i)), hold on
    xlabel('time'), ylabel('pull')
    H3 = subplot(3,1,3); plot(t,D(i,:),colour(i)), hold on
    xlabel('time'), ylabel('fluid output')

    tags = [tags; num2str(i)];
end

legend(H1,tags,'Location','NorthWest')
legend(H2,tags,'Location','NorthWest')
legend(H3,tags,'Location','NorthWest')
```

Appendix B ODEs for a system with three machines

In this appendix the ordinary differential equations are presented for a system with three machines. The ODEs are used in section 3.2 to analyze stability and create the modes graphs.

Case 2 : $\rho_3 > 1, \rho_{1,2} < 1$

Mode 1 – $\{0,0,0\}$

$\dot{Y}_1 = \theta_1\rho_1 - (1-\theta_2) = 0$

$\dot{Y}_2 = \theta_2\rho_2 - (1-\theta_3) = 0$

$\dot{Y}_3 = \theta_3\rho_3 - (1-\theta_1) = 0$

Mode 2 – $\{0,0,1\}$

$\dot{Y}_1 = 0$

$\dot{Y}_2 = 0$

$\dot{Y}_3 = \rho_3(1-\rho_2) - 1 < 0$

Mode 3 – $\{0,1,0\}$

$\dot{Y}_1 = 0$

$\dot{Y}_2 = \rho_2(1-\rho_1) - 1 < 0$

$\dot{Y}_3 = 0$

Mode 4 – $\{0,1,1\}$

$\dot{Y}_1 = 0$

$\dot{Y}_2 = \rho_2 - 1 < 0$

$\dot{Y}_3 = -1 < 0$

Mode 5 – $\{1,0,0\}$

$\dot{Y}_1 = -1 < 0$

$\dot{Y}_2 = 0$

$\dot{Y}_3 = \rho_3 - 1 > 0$

Mode 6 – $\{1,0,1\}$

$\dot{Y}_1 = -1 < 0$

$\dot{Y}_2 = 0$

$\dot{Y}_3 = \rho_3 - 1 > 0$

Mode 7 – $\{1,1,0\}$

$\dot{Y}_1 = \rho_1 - 1 < 0$

$\dot{Y}_2 = -1 < 0$

$\dot{Y}_3 = 0$

Mode 8 – $\{1,1,1\}$

$\dot{Y}_1 = -1 < 0$

$\dot{Y}_2 = -1 < 0$

$\dot{Y}_3 = -1 < 0$

Case 3 : $\rho_1 < 1, \rho_{2,3} > 1$

Mode 1 – $\{0,0,0\}$

$\dot{Y_1} = \theta_1\rho_1 - (1-\theta_2) = 0$

$\dot{Y_2} = \theta_2\rho_2 - (1-\theta_3) = 0$

$\dot{Y_3} = \theta_3\rho_3 - (1-\theta_1) = 0$

Mode 2 – $\{0,0,1\}$

$\dot{Y_1} = 0$

$\dot{Y_2} = \rho_2 - 1 > 0$

$\dot{Y_3} = -1 < 0$

Mode 3 – $\{0,1,0\}$

$\dot{Y_1} = 0$

$\dot{Y_2} = \rho_2(1-\rho_1) - 1 < 0$

$\dot{Y_3} = 0$

Mode 4 – $\{0,1,1\}$

$\dot{Y_1} = 0$

$\dot{Y_2} = \rho_2 - 1 > 0$

$\dot{Y_3} = -1 < 0$

Mode 5 – $\{1,0,0\}$

$\dot{Y_1} = -1 < 0$

$\dot{Y_2} = 0$

$\dot{Y_3} = \rho_3 - 1 > 0$

Mode 6 – $\{1,0,1\}$

$\dot{Y_1} = -1 < 0$

$\dot{Y_2} = 0$

$\dot{Y_3} = \rho_3 - 1 > 0$

Mode 7 – $\{1,1,0\}$

$\dot{Y_1} = \rho_1 - 1 < 0$

$\dot{Y_2} = -1 < 0$

$\dot{Y_3} = 0$

Mode 8 – $\{1,1,1\}$

$\dot{Y_1} = -1 < 0$

$\dot{Y_2} = -1 < 0$

$\dot{Y_3} = -1 < 0$

Case 4 : $\rho_i > 1$

Mode 1 – $\{0,0,0\}$

$\dot{Y}_1 = \theta_1\rho_1 - (1-\theta_2) = 0$

$\dot{Y}_2 = \theta_2\rho_2 - (1-\theta_3) = 0$

$\dot{Y}_3 = \theta_3\rho_3 - (1-\theta_1) = 0$

Mode 2 – $\{0,0,1\}$

$\dot{Y}_1 = 0$

$\dot{Y}_2 = \rho_2 - 1 > 0$

$\dot{Y}_3 = -1 < 0$

Mode 3 – $\{0,1,0\}$

$\dot{Y}_1 = \rho_1 - 1 > 0$

$\dot{Y}_2 = -1 < 0$

$\dot{Y}_3 = 0$

Mode 4 – $\{0,1,1\}$

$\dot{Y}_1 = 0$

$\dot{Y}_2 = \rho_2 - 1 > 0$

$\dot{Y}_3 = -1 < 0$

Mode 5 – $\{1,0,0\}$

$\dot{Y}_1 = -1 < 0$

$\dot{Y}_2 = 0$

$\dot{Y}_3 = \rho_3 - 1 > 0$

Mode 6 – $\{1,0,1\}$

$\dot{Y}_1 = -1 < 0$

$\dot{Y}_2 = 0$

$\dot{Y}_3 = \rho_3 - 1 > 0$

Mode 7 – $\{1,1,0\}$

$\dot{Y}_1 = \rho_1 - 1 > 0$

$\dot{Y}_2 = -1 < 0$

$\dot{Y}_3 = 0$

Mode 8 – $\{1,1,1\}$

$\dot{Y}_1 = -1 < 0$

$\dot{Y}_2 = -1 < 0$

$\dot{Y}_3 = -1 < 0$

Appendix C ODEs for a system with four machines

In this appendix the ordinary differential equations are presented for a system with four machines. The ODEs are used in section 3.2 to analyze stability and create the modes graphs.

Case 2 : $\rho_1 > 1, \rho_{2,3,4} < 1$

Mode 1 - $\{0,0,0,0\}$

$$\dot{Y}_1 = 0$$

$$\dot{Y}_2 = 0$$

$$\dot{Y}_3 = 0$$

$$\dot{Y}_4 = 0$$

Mode 2 - $\{0,0,0,1\}$

$$\dot{Y}_1 = 0$$

$$\dot{Y}_2 = 0$$

$$\dot{Y}_3 = 0$$

$$\dot{Y}_4 = (1-\rho_3(1-\rho_2)\rho_4 - 1 < 0$$

Mode 3 - $\{0,0,1,0\}$

$$\dot{Y}_1 = \rho_1 - 1 > 0$$

$$\dot{Y}_2 = 0$$

$$\dot{Y}_3 = \rho_3 - 1 < 0$$

$$\dot{Y}_4 = 0$$

Mode 4 - $\{0,0,1,1\}$

$$\dot{Y}_1 = 0$$

$$\dot{Y}_2 = 0$$

$$\dot{Y}_3 = (1-\rho_2)\rho_3 - 1 < 0$$

$$\dot{Y}_4 = -1 < 0$$

Mode 5 - $\{0,1,0,0\}$

$$\dot{Y}_1 = 0$$

$$\dot{Y}_2 = (1-\rho_1(1-\rho_4))\rho_2 - 1 < 0$$

$$\dot{Y}_3 = 0$$

$$\dot{Y}_4 = 0$$

Mode 6 - $\{0,1,0,1\}$

$$\dot{Y}_1 = 0$$

$$\dot{Y}_2 = \rho_2 - 1 < 0$$

$$\dot{Y}_3 = 0$$

$$\dot{Y}_4 = \rho_4 - 1 < 0$$

Mode 7 - $\{0,1,1,0\}$

$$\dot{Y}_1 = \rho_1 - 1 > 0$$

$$\dot{Y}_2 = -1 < 0$$

$$\dot{Y}_3 = -1 < 0$$

$$\dot{Y}_4 = 0$$

Mode 8 - $\{0,1,1,1\}$

$$\dot{Y}_1 = 0$$

$$\dot{Y}_2 = \rho_2 - 1 < 0$$

$$\dot{Y}_3 = -1 < 0$$

$$\dot{Y}_4 = -1 < 0$$

Mode 9 - $\{1,0,0,0\}$

$$\dot{Y}_1 = (1-\rho_4(1-\rho_3))\rho_1 - 1 < 0$$
$$\dot{Y}_2 = 0$$
$$\dot{Y}_3 = 0$$
$$\dot{Y}_4 = 0$$

Mode 10 - $\{1,0,0,1\}$

$$\dot{Y}_1 = -1 < 0$$
$$\dot{Y}_2 = 0$$
$$\dot{Y}_3 = 0$$
$$\dot{Y}_4 = (1-\rho_3)\rho_4 - 1 < 0$$

Mode 11 - $\{1,0,1,0\}$

$$\dot{Y}_1 = \rho_1 - 1 > 0$$
$$\dot{Y}_2 = 0$$
$$\dot{Y}_3 = \rho_3 - 1 < 0$$
$$\dot{Y}_4 = 0$$

Mode 12 - $\{1,0,1,1\}$

$$\dot{Y}_1 = -1 < 0$$
$$\dot{Y}_2 = 0$$
$$\dot{Y}_3 = \rho_3 - 1 < 0$$
$$\dot{Y}_4 = -1 < 0$$

Mode 13 - $\{1,1,0,0\}$

$$\dot{Y}_1 = (1-\rho_4)\rho_1 - 1 < 0$$
$$\dot{Y}_2 = -1 < 0$$
$$\dot{Y}_3 = 0$$
$$\dot{Y}_4 = 0$$

Mode 14 - $\{1,1,0,1\}$

$$\dot{Y}_1 = -1 < 0$$
$$\dot{Y}_2 = -1 < 0$$
$$\dot{Y}_3 = 0$$
$$\dot{Y}_4 = \rho_4 - 1 < 0$$

Mode 15 - $\{1,1,1,0\}$

$$\dot{Y}_1 = \rho_1 - 1 > 0$$
$$\dot{Y}_2 = -1 < 0$$
$$\dot{Y}_3 = -1 < 0$$
$$\dot{Y}_4 = 0$$

Mode 16 - $\{1,1,1,1\}$

$$\dot{Y}_1 = -1 < 0$$
$$\dot{Y}_2 = -1 < 0$$
$$\dot{Y}_3 = -1 < 0$$
$$\dot{Y}_4 = -1 < 0$$

Case 3a : $\rho_{1,2} > 1, \rho_{3,4} < 1$

Mode 1 - $\{0,0,0,0\}$

$\dot{Y}_1 = 0$

$\dot{Y}_2 = 0$

$\dot{Y}_3 = 0$

$\dot{Y}_4 = 0$

Mode 2 - $\{0,0,0,1\}$

$\dot{Y}_1 = 0$

$\dot{Y}_2 = \rho_2 - 1 > 0$

$\dot{Y}_3 = 0$

$\dot{Y}_4 = \rho_4 - 1 < 0$

Mode 3 - $\{0,0,1,0\}$

$\dot{Y}_1 = \rho_1 - 1 > 0$

$\dot{Y}_2 = 0$

$\dot{Y}_3 = \rho_3 - 1 < 0$

$\dot{Y}_4 = 0$

Mode 4 - $\{0,0,1,1\}$

$\dot{Y}_1 = 0$

$\dot{Y}_2 = \rho_2 - 1 > 0$

$\dot{Y}_3 = -1 < 0$

$\dot{Y}_4 = -1 < 0$

Mode 5 - $\{0,1,0,0\}$

$\dot{Y}_1 = 0$

$\dot{Y}_2 = (1 - \rho_1(1 - \rho_4))\rho_2 - 1 < 0$

$\dot{Y}_3 = 0$

$\dot{Y}_4 = 0$

Mode 6 - $\{0,1,0,1\}$

$\dot{Y}_1 = 0$

$\dot{Y}_2 = \rho_2 - 1 > 0$

$\dot{Y}_3 = 0$

$\dot{Y}_4 = \rho_4 - 1 < 0$

Mode 7 - $\{0,1,1,0\}$

$\dot{Y}_1 = \rho_1 - 1 > 0$

$\dot{Y}_2 = -1 < 0$

$\dot{Y}_3 = -1 < 0$

$\dot{Y}_4 = 0$

Mode 8 - $\{0,1,1,1\}$

$\dot{Y}_1 = 0$

$\dot{Y}_2 = \rho_2 - 1 > 0$

$\dot{Y}_3 = -1 < 0$

$\dot{Y}_4 = -1 < 0$

Mode 9 - $\{1,0,0,0\}$

$$\dot{Y}_1 = (1-\rho_4(1-\rho_3))\rho_1 - 1 < 0$$
$$\dot{Y}_2 = 0$$
$$\dot{Y}_3 = 0$$
$$\dot{Y}_4 = 0$$

Mode 10 - $\{1,0,0,1\}$

$$\dot{Y}_1 = -1 < 0$$
$$\dot{Y}_2 = 0$$
$$\dot{Y}_3 = 0$$
$$\dot{Y}_4 = (1-\rho_3)\rho_4 - 1 < 0$$

Mode 11 - $\{1,0,1,0\}$

$$\dot{Y}_1 = \rho_1 - 1 > 0$$
$$\dot{Y}_2 = 0$$
$$\dot{Y}_3 = \rho_3 - 1 < 0$$
$$\dot{Y}_4 = 0$$

Mode 12 - $\{1,0,1,1\}$

$$\dot{Y}_1 = -1 < 0$$
$$\dot{Y}_2 = 0$$
$$\dot{Y}_3 = \rho_3 - 1 < 0$$
$$\dot{Y}_4 = -1 < 0$$

Mode 13 - $\{1,1,0,0\}$

$$\dot{Y}_1 = (1-\rho_4)\rho_1 - 1 < 0$$
$$\dot{Y}_2 = -1 < 0$$
$$\dot{Y}_3 = 0$$
$$\dot{Y}_4 = 0$$

Mode 14 - $\{1,1,0,1\}$

$$\dot{Y}_1 = -1 < 0$$
$$\dot{Y}_2 = -1 < 0$$
$$\dot{Y}_3 = 0$$
$$\dot{Y}_4 = \rho_4 - 1 < 0$$

Mode 15 - $\{1,1,1,0\}$

$$\dot{Y}_1 = \rho_1 - 1 > 0$$
$$\dot{Y}_2 = -1 < 0$$
$$\dot{Y}_3 = -1 < 0$$
$$\dot{Y}_4 = 0$$

Mode 16 - $\{1,1,1,1\}$

$$\dot{Y}_1 = -1 < 0$$
$$\dot{Y}_2 = -1 < 0$$
$$\dot{Y}_3 = -1 < 0$$
$$\dot{Y}_4 = -1 < 0$$

Case 3b : $\rho_{1,2} > 1, \rho_{3,4} < 1$

Mode 1 - $\{0,0,0,0\}$

$\dot{Y}_1 = 0$

$\dot{Y}_2 = 0$

$\dot{Y}_3 = 0$

$\dot{Y}_4 = 0$

Mode 2 - $\{0,0,0,1\}$

$\dot{Y}_1 = 0$

$\dot{Y}_2 = \rho_2 - 1 > 0$

$\dot{Y}_3 = 0$

$\dot{Y}_4 = \rho_4 - 1 < 0$

Mode 3 - $\{0,0,1,0\}$

$\dot{Y}_1 = \rho_1 - 1 > 0$

$\dot{Y}_2 = 0$

$\dot{Y}_3 = \rho_3 - 1 < 0$

$\dot{Y}_4 = 0$

Mode 4 - $\{0,0,1,1\}$

$\dot{Y}_1 = 0$

$\dot{Y}_2 = \rho_2 - 1 > 0$

$\dot{Y}_3 = -1 < 0$

$\dot{Y}_4 = -1 < 0$

Mode 5 - $\{0,1,0,0\}$

$\dot{Y}_1 = 0$

$\dot{Y}_2 = (1 - \rho_1(1 - \rho_4))\rho_2 - 1 < 0$

$\dot{Y}_3 = 0$

$\dot{Y}_4 = 0$

Mode 6 - $\{0,1,0,1\}$

$\dot{Y}_1 = 0$

$\dot{Y}_2 = \rho_2 - 1 > 0$

$\dot{Y}_3 = 0$

$\dot{Y}_4 = \rho_4 - 1 < 0$

Mode 7 - $\{0,1,1,0\}$

$\dot{Y}_1 = \rho_1 - 1 > 0$

$\dot{Y}_2 = -1 < 0$

$\dot{Y}_3 = -1 < 0$

$\dot{Y}_4 = 0$

Mode 8 - $\{0,1,1,1\}$

$\dot{Y}_1 = 0$

$\dot{Y}_2 = \rho_2 - 1 > 0$

$\dot{Y}_3 = -1 < 0$

$\dot{Y}_4 = -1 < 0$

Mode 9 - $\{1,0,0,0\}$

$$\dot{Y}_1 = (1-\rho_4(1-\rho_3))\rho_1 - 1 > 0$$
$$\dot{Y}_2 = 0$$
$$\dot{Y}_3 = 0$$
$$\dot{Y}_4 = 0$$

Mode 10 - $\{1,0,0,1\}$

$$\dot{Y}_1 = -1 < 0$$
$$\dot{Y}_2 = 0$$
$$\dot{Y}_3 = 0$$
$$\dot{Y}_4 = (1-\rho_3)\rho_4 - 1 < 0$$

Mode 11 - $\{1,0,1,0\}$

$$\dot{Y}_1 = \rho_1 - 1 > 0$$
$$\dot{Y}_2 = 0$$
$$\dot{Y}_3 = \rho_3 - 1 < 0$$
$$\dot{Y}_4 = 0$$

Mode 12 - $\{1,0,1,1\}$

$$\dot{Y}_1 = -1 < 0$$
$$\dot{Y}_2 = 0$$
$$\dot{Y}_3 = \rho_3 - 1 < 0$$
$$\dot{Y}_4 = -1 < 0$$

Mode 13 - $\{1,1,0,0\}$

$$\dot{Y}_1 = (1-\rho_4)\rho_1 - 1 > 0$$
$$\dot{Y}_2 = -1 < 0$$
$$\dot{Y}_3 = 0$$
$$\dot{Y}_4 = 0$$

Mode 14 - $\{1,1,0,1\}$

$$\dot{Y}_1 = -1 < 0$$
$$\dot{Y}_2 = -1 < 0$$
$$\dot{Y}_3 = 0$$
$$\dot{Y}_4 = \rho_4 - 1 < 0$$

Mode 15 - $\{1,1,1,0\}$

$$\dot{Y}_1 = \rho_1 - 1 > 0$$
$$\dot{Y}_2 = -1 < 0$$
$$\dot{Y}_3 = -1 < 0$$
$$\dot{Y}_4 = 0$$

Mode 16 - $\{1,1,1,1\}$

$$\dot{Y}_1 = -1 < 0$$
$$\dot{Y}_2 = -1 < 0$$
$$\dot{Y}_3 = -1 < 0$$
$$\dot{Y}_4 = -1 < 0$$

Case 5 : $\rho_{1,2,3} > 1, \rho_4 < 1$

Mode 1 - $\{0,0,0,0\}$

$\dot{Y}_1 = 0$

$\dot{Y}_2 = 0$

$\dot{Y}_3 = 0$

$\dot{Y}_4 = 0$

Mode 2 - $\{0,0,0,1\}$

$\dot{Y}_1 = 0$

$\dot{Y}_2 = \rho_2 - 1 > 0$

$\dot{Y}_3 = 0$

$\dot{Y}_4 = \rho_4 - 1 < 0$

Mode 3 - $\{0,0,1,0\}$

$\dot{Y}_1 = \rho_1 - 1 > 0$

$\dot{Y}_2 = 0$

$\dot{Y}_3 = \rho_3 - 1 > 0$

$\dot{Y}_4 = 0$

Mode 4 - $\{0,0,1,1\}$

$\dot{Y}_1 = 0$

$\dot{Y}_2 = \rho_2 - 1 > 0$

$\dot{Y}_3 = -1 < 0$

$\dot{Y}_4 = -1 < 0$

Mode 5 - $\{0,1,0,0\}$

$\dot{Y}_1 = 0$

$\dot{Y}_2 = (1 - \rho_1(1 - \rho_4))\rho_2 - 1 > 0$

$\dot{Y}_3 = 0$

$\dot{Y}_4 = 0$

Mode 6 - $\{0,1,0,1\}$

$\dot{Y}_1 = 0$

$\dot{Y}_2 = \rho_2 - 1 > 0$

$\dot{Y}_3 = 0$

$\dot{Y}_4 = \rho_4 - 1 < 0$

Mode 7 - $\{0,1,1,0\}$

$\dot{Y}_1 = \rho_1 - 1 > 0$

$\dot{Y}_2 = -1 < 0$

$\dot{Y}_3 = -1 < 0$

$\dot{Y}_4 = 0$

Mode 8 - $\{0,1,1,1\}$

$\dot{Y}_1 = 0$

$\dot{Y}_2 = \rho_2 - 1 > 0$

$\dot{Y}_3 = -1 < 0$

$\dot{Y}_4 = -1 < 0$

Mode 9 - $\{1,0,0,0\}$

$$\dot{Y}_1 = \rho_1 - 1 > 0$$

$$\dot{Y}_2 = 0$$

$$\dot{Y}_3 = \rho_3 - 1 > 0$$

$$\dot{Y}_4 = 0$$

Mode 10 - $\{1,0,0,1\}$

$$\dot{Y}_1 = -1 < 0$$

$$\dot{Y}_2 = 0$$

$$\dot{Y}_3 = \rho_3 - 1 > 0$$

$$\dot{Y}_4 = -1 < 0$$

Mode 11 - $\{1,0,1,0\}$

$$\dot{Y}_1 = \rho_1 - 1 > 0$$

$$\dot{Y}_2 = 0$$

$$\dot{Y}_3 = \rho_3 - 1 > 0$$

$$\dot{Y}_4 = 0$$

Mode 12 - $\{1,0,1,1\}$

$$\dot{Y}_1 = -1 < 0$$

$$\dot{Y}_2 = 0$$

$$\dot{Y}_3 = \rho_3 - 1 > 0$$

$$\dot{Y}_4 = -1 < 0$$

Mode 13 - $\{1,1,0,0\}$

$$\dot{Y}_1 = (1-\rho_4)\rho_1 - 1 < 0$$

$$\dot{Y}_2 = -1 < 0$$

$$\dot{Y}_3 = 0$$

$$\dot{Y}_4 = 0$$

Mode 14 - $\{1,1,0,1\}$

$$\dot{Y}_1 = -1 < 0$$

$$\dot{Y}_2 = -1 < 0$$

$$\dot{Y}_3 = 0$$

$$\dot{Y}_4 = \rho_4 - 1 < 0$$

Mode 15 - $\{1,1,1,0\}$

$$\dot{Y}_1 = \rho_1 - 1 > 0$$

$$\dot{Y}_2 = -1 < 0$$

$$\dot{Y}_3 = -1 < 0$$

$$\dot{Y}_4 = 0$$

Mode 16 - $\{1,1,1,1\}$

$$\dot{Y}_1 = -1 < 0$$

$$\dot{Y}_2 = -1 < 0$$

$$\dot{Y}_3 = -1 < 0$$

$$\dot{Y}_4 = -1 < 0$$

Case 6 : $\rho_i > 1$

Mode 1 - $\{0,0,0,0\}$

$\dot{Y}_1 = 0$

$\dot{Y}_2 = 0$

$\dot{Y}_3 = 0$

$\dot{Y}_4 = 0$

Mode 2 - $\{0,0,0,1\}$

$\dot{Y}_1 = 0$

$\dot{Y}_2 = \rho_2 - 1 > 0$

$\dot{Y}_3 = 0$

$\dot{Y}_4 = \rho_4 - 1 > 0$

Mode 3 - $\{0,0,1,0\}$

$\dot{Y}_1 = \rho_1 - 1 > 0$

$\dot{Y}_2 = 0$

$\dot{Y}_3 = \rho_3 - 1 > 0$

$\dot{Y}_4 = 0$

Mode 4 - $\{0,0,1,1\}$

$\dot{Y}_1 = 0$

$\dot{Y}_2 = \rho_2 - 1 > 0$

$\dot{Y}_3 = -1 < 0$

$\dot{Y}_4 = -1 < 0$

Mode 5 - $\{0,1,0,0\}$

$\dot{Y}_1 = 0$

$\dot{Y}_2 = \rho_2 - 1 > 0$

$\dot{Y}_3 = 0$

$\dot{Y}_4 = \rho_4 - 1 > 0$

Mode 6 - $\{0,1,0,1\}$

$\dot{Y}_1 = 0$

$\dot{Y}_2 = \rho_2 - 1 > 0$

$\dot{Y}_3 = 0$

$\dot{Y}_4 = \rho_4 - 1 > 0$

Mode 7 - $\{0,1,1,0\}$

$\dot{Y}_1 = \rho_1 - 1 > 0$

$\dot{Y}_2 = -1 < 0$

$\dot{Y}_3 = -1 < 0$

$\dot{Y}_4 = 0$

Mode 8 - $\{0,1,1,1\}$

$\dot{Y}_1 = 0$

$\dot{Y}_2 = \rho_2 - 1 > 0$

$\dot{Y}_3 = -1 < 0$

$\dot{Y}_4 = -1 < 0$

Mode 9 - $\{1,0,0,0\}$

$\dot{Y}_1 = \rho_1 - 1 > 0$

$\dot{Y}_2 = 0$

$\dot{Y}_3 = \rho_3 - 1 > 0$

$\dot{Y}_4 = 0$

Mode 10 - $\{1,0,0,1\}$

$\dot{Y}_1 = -1 < 0$

$\dot{Y}_2 = 0$

$\dot{Y}_3 = \rho_3 - 1 > 0$

$\dot{Y}_4 = -1 < 0$

Mode 11 - $\{1,0,1,0\}$

$\dot{Y}_1 = \rho_1 - 1 > 0$

$\dot{Y}_2 = 0$

$\dot{Y}_3 = \rho_3 - 1 > 0$

$\dot{Y}_4 = 0$

Mode 12 - $\{1,0,1,1\}$

$\dot{Y}_1 = -1 < 0$

$\dot{Y}_2 = 0$

$\dot{Y}_3 = \rho_3 - 1 > 0$

$\dot{Y}_4 = -1 < 0$

Mode 13 - $\{1,1,0,0\}$

$\dot{Y}_1 = -1 < 0$

$\dot{Y}_2 = -1 < 0$

$\dot{Y}_3 = 0$

$\dot{Y}_4 = \rho_4 - 1 > 0$

Mode 14 - $\{1,1,0,1\}$

$\dot{Y}_1 = -1 < 0$

$\dot{Y}_2 = -1 < 0$

$\dot{Y}_3 = 0$

$\dot{Y}_4 = \rho_4 - 1 > 0$

Mode 15 - $\{1,1,1,0\}$

$\dot{Y}_1 = \rho_1 - 1 > 0$

$\dot{Y}_2 = -1 < 0$

$\dot{Y}_3 = -1 < 0$

$\dot{Y}_4 = 0$

Mode 16 - $\{1,1,1,1\}$

$\dot{Y}_1 = -1 < 0$

$\dot{Y}_2 = -1 < 0$

$\dot{Y}_3 = -1 < 0$

$\dot{Y}_4 = -1 < 0$

Appendix D Proof for stability for three machines

In section 3.2 four cases for $M = 3$ are analyzed. One of the requirements for a system to be stable is that all $0 < \theta_i < 1$. With the use of some restrictions on ρ_i this can be achieved. This appendix focuses on proving that these restrictions fulfil the requirement of $0 < \theta_i < 1$. (3.2) is presented once again below:

$$\theta_1 = \frac{1-(1-\rho_2)\rho_3}{1+\rho_1\rho_2\rho_3},\ \theta_1 > 0 \ \mathit{iff}\ 1-(1-\rho_2)\rho_3 > 0$$
$$\theta_2 = \frac{1-(1-\rho_3)\rho_1}{1+\rho_1\rho_2\rho_3},\ \theta_2 > 0 \ \mathit{iff}\ 1-(1-\rho_3)\rho_1 > 0$$
$$\theta_3 = \frac{1-(1-\rho_1)\rho_2}{1+\rho_1\rho_2\rho_3},\ \theta_3 > 0 \ \mathit{iff}\ 1-(1-\rho_1)\rho_2 > 0$$

which can be simplified to $\theta_i > 0 \ \mathit{iff}\ 1-(1-\rho_{i+1})\rho_{i+2} > 0$.

Case 1 : $\rho_i < 1$

There are no additional restrictions on ρ_i. The proof for all θ_i is the same, therefore only $i = 1$ is done.

$$\begin{aligned} 0 &< 1-\rho_2 < 1 \\ 0 &< (1-\rho_2)\rho_3 < 1 \\ 0 &> -(1-\rho_2)\rho_3 > -1 \\ 1 &> 1-(1-\rho_2)\rho_3 > 0 \end{aligned}$$

This proves the system behaves in a stable manner if the first mode is reached.

Case 2 : $\rho_3 > 1, \rho_{1,2} < 1$

One additional restriction is used to get all θ_i positive, this restriction is $\rho_3 < 1/(1-\rho_2)$. The proof is split into three parts, each part concerns one of the three θ_i.

The restriction is designed to make θ_1 positive, the proof is presented below.

$$\begin{aligned} 0 &< 1-\rho_2 < 1 \\ 0 &< (1-\rho_2)\rho_3 < 1 \\ 0 &> -(1-\rho_2)\rho_3 > -1 \\ 1 &> 1-(1-\rho_2)\rho_3 > 0 \end{aligned}$$

θ_2 is also positive,

$$1 < \rho_3 < \frac{1}{1-\rho_2}$$
$$0 > (1-\rho_3) > \frac{-\rho_2}{1-\rho_2}$$
$$0 < -(1-\rho_3)\rho_1 < \frac{\rho_2\rho_1}{1-\rho_2}$$
$$1 < 1-(1-\rho_3)\rho_1 < 1+\frac{\rho_2\rho_1}{1-\rho_2}$$

This proves θ_2 is positive. The last proof is identical to the one presented in case 1 above, so θ_3 is positive as well.

As all θ_i are positive, they are also between 0 and 1, this was first proven in section 3.1.

Case 3 : $\rho_1 < 1, \rho_{2,3} > 1$

The restriction that is used is that $\rho_2 < 1/(1-\rho_1)$. The proof for stability is once again split into three parts, starting with θ_1.

Presented below is the proof for the first θ.

$$1-\rho_2 < 0$$
$$(1-\rho_2)\rho_3 < 0$$
$$-(1-\rho_2)\rho_3 > 0$$
$$1-(1-\rho_2)\rho_3 > 1$$

The proof for θ_2 is identical to the one above

$$1-\rho_3 < 0$$
$$(1-\rho_3)\rho_1 < 0$$
$$-(1-\rho_3)\rho_1 > 0$$
$$1-(1-\rho_3)\rho_1 > 1$$

The restriction is introduced so that θ_3 is also possible

$$
\begin{aligned}
&0 < 1-\rho_1 < 1 \\
&0 < (1-\rho_1)\rho_2 < 1 \\
&-1 > -(1-\rho_1)\rho_2 > 0 \\
&0 > 1-(1-\rho_1)\rho_2 > 1
\end{aligned}
$$

This proves that all θ_i are positive and therefore $0 < \theta_i < 1$, this system will behave in a stable manner once it reaches the mode in which all queues are empty.

Case 4 : $\rho_i > 1$

There are no additional requirements needed for this case. The proof for all θ_i is the same, therefore only $i = 1$ is done.

$$
\begin{aligned}
&1-\rho_2 < 0 \\
&(1-\rho_2)\rho_3 < 0 \\
&-(1-\rho_2)\rho_3 > 0 \\
&1-(1-\rho_2)\rho_3 > 1
\end{aligned}
$$

This proves the system behaves in a stable manner if the first mode is reached.

Appendix E Proof for (in)stability for four machines

In section 3.3 six cases for $M = 4$ are analyzed. One of the requirements for a system to be stable is that all $0 < \theta_i < 1$. With the use of some restrictions on ρ_i this can be achieved. This appendix focuses on proving that these restrictions fulfil the requirement of $0 < \theta_i < 1$. (3.4) is presented once again below:

$$\theta_1 = \frac{\rho_2\rho_3\rho_4 - \rho_3\rho_4 + \rho_4 - 1}{\rho_1\rho_2\rho_3\rho_4 - 1} = \frac{\rho_4(1-(1-\rho_2)\rho_3) - 1}{\rho_1\rho_2\rho_3\rho_4 - 1}$$
$$\theta_2 = \frac{\rho_1\rho_3\rho_4 - \rho_1\rho_4 + \rho_1 - 1}{\rho_1\rho_2\rho_3\rho_4 - 1} = \frac{\rho_1(1-(1-\rho_3)\rho_4) - 1}{\rho_1\rho_2\rho_3\rho_4 - 1}$$
$$\theta_3 = \frac{\rho_1\rho_2\rho_4 - \rho_1\rho_2 + \rho_2 - 1}{\rho_1\rho_2\rho_3\rho_4 - 1} = \frac{\rho_2(1-(1-\rho_4)\rho_1) - 1}{\rho_1\rho_2\rho_3\rho_4 - 1}$$
$$\theta_4 = \frac{\rho_1\rho_2\rho_3 - \rho_2\rho_3 + \rho_3 - 1}{\rho_1\rho_2\rho_3\rho_4 - 1} = \frac{\rho_3(1-(1-\rho_1)\rho_2) - 1}{\rho_1\rho_2\rho_3\rho_4 - 1}$$

Case 2 : $\rho_1 > 1, \rho_{2,3,4} < 1$

First, we prove the numerator of θ_1 is always negative.

$$0 < 1 - \rho_2 < 1$$
$$0 < (1-\rho_2)\rho_3 < 1$$
$$1 > 1 - (1-\rho_2)\rho_3 > 0$$
$$1 > \rho_4(1-(1-\rho_2)\rho_3) > 0$$
$$0 > \rho_4(1-(1-\rho_2)\rho_3) - 1 > -1$$

The restriction $\rho_1 < 1/(1-\rho_4(1-\rho_3))$ makes the numerator of θ_2 negative.

$$0 < 1 - \rho_3 < 1$$
$$0 < (1-\rho_3)\rho_4 < 1$$
$$1 > 1 - (1-\rho_3)\rho_4 > 0$$
$$1 > \rho_1(1-(1-\rho_3)\rho_4) > 0$$
$$0 > \rho_1(1-(1-\rho_3)\rho_4) - 1 > -1$$

The numerator of θ_3 is always negative.

$$
\begin{aligned}
&0 < 1-\rho_4 < 1 \\
&0 < (1-\rho_4)\rho_1 < \rho_1 \\
&1 > 1-(1-\rho_4)\rho_1 > 1-\rho_1 \\
&\rho_2 > \rho_2(1-(1-\rho_4)\rho_1) > \rho_2(1-\rho_1) \\
&\rho_2 - 1 > \rho_2(1-(1-\rho_4)\rho_1) - 1 > \rho_2(1-\rho_1) - 1
\end{aligned}
$$

However, the expression $1-\rho_1(1-\rho_4)$ has to be positive or else it will conflict with the other restrictions.

Using a similar analysis as for θ_2, the restriction $\rho_1 < (1-\rho_3(1-\rho_2))/(\rho_2\rho_3)$ makes the numerator of θ_4 negative.

Case 3a : $\rho_{1,2} > 1, \rho_{3,4} < 1$ and $\rho_1\rho_2\rho_3\rho_4 < 1$

Because $\rho_1\rho_2\rho_3\rho_4 < 1$, the denominator is negative and all numerators have to be negative. It is proven that the numerators can be both positive and negative, with the use of the correct requirements all $\theta_i > 0$ I.e. a correct requirement for θ_1 would be that $\rho_4 < \rho_4(1-(1-\rho_2)\rho_3) < 1$.

First, the numerator of θ_1 is analyzed.

$$
\begin{aligned}
&1-\rho_2 < 0 \\
&(1-\rho_2)\rho_3 < 0 \\
&1-(1-\rho_2)\rho_3 > 1 \\
&\rho_4(1-(1-\rho_2)\rho_3) > \rho_4 \\
&\rho_4(1-(1-\rho_2)\rho_3) - 1 > \rho_4 - 1
\end{aligned}
$$

which can be positive or negative as $\rho_4 < 1$. The second fraction can be positive or negative as well.

$$
\begin{aligned}
&0 < 1-\rho_3 < 1 \\
&0 < (1-\rho_3)\rho_4 < 1 \\
&1 > 1-(1-\rho_3)\rho_4 > 0 \\
&\rho_1 > \rho_1(1-(1-\rho_3)\rho_4) > 0 \\
&\rho_1 - 1 > \rho_1(1-(1-\rho_3)\rho_4) > -1
\end{aligned}
$$

As $\rho_1 > 1$ the numerator is positive or negative. A similar analysis is done for θ_3,

$$0 < 1 - \rho_4 < 1$$
$$0 < (1 - \rho_4)\rho_1 < \rho_1$$
$$1 > 1 - (1 - \rho_4)\rho_1 > 1 - \rho_1$$

The restriction states that $1 - (1 - \rho_4)\rho_1 > 0$, so we continue the analysis with this boundary.

$$1 > 1 - (1 - \rho_4)\rho_1 > 0$$
$$\rho_2 > \rho_2(1 - (1 - \rho_4)\rho_1) > 0$$
$$\rho_2 - 1 > \rho_2(1 - (1 - \rho_4)\rho_1) - 1 > -1$$

This proves that, with the restriction, the numerator of θ_3 can be both positive or negative.

The last proof can be found below.

$$1 - \rho_1 < 0$$
$$(1 - \rho_1)\rho_2 < 0$$
$$1 - (1 - \rho_1)\rho_2 > 1$$
$$\rho_3(1 - (1 - \rho_1)\rho_2) > \rho_3$$
$$\rho_3(1 - (1 - \rho_1)\rho_2) - 1 > \rho_3 - 1$$

Which can be either positive or negative as $\rho_3 < 1$.

Case 3b : $\rho_{1,2} > 1, \rho_{3,4} < 1$ and $\rho_1\rho_2\rho_3\rho_4 > 1$

The proof for case 3b can be found above, see case 3a. For that case, it is shown that all numerators of θ_i can be both positive and negative. With the use of the opposing requirements for θ_i as in case 3a and together with $1 - (1 - \rho_4)\rho_1 > 0$ this makes all θ_i positive.

Case 4 : $\rho_{1,3} > 1, \rho_{2,4} < 1$

This case is always unstable as the numerator of θ_1 is always negative and the numerator of θ_2 is always positive. First θ_1 is examined.

$$
\begin{aligned}
&0 < 1-\rho_2 < 1 \\
&0 < (1-\rho_2)\rho_3 < \rho_3 \\
&0 > -(1-\rho_2)\rho_3 > -\rho_3 \\
&1 > 1-(1-\rho_2)\rho_3 > 1-\rho_3 \\
&1 > \rho_4(1-(1-\rho_2)\rho_3) > \rho_4(1-\rho_3) \\
&0 > \rho_4(1-(1-\rho_2)\rho_3)-1 > \rho_4(1-\rho_3)-1
\end{aligned}
$$

It can be seen that the numerator of θ_1 is always negative. For θ_2 it can be proven that it is always positive.

$$
\begin{aligned}
&1-\rho_3 < 0 \\
&(1-\rho_3)\rho_4 < 0 \\
&1-(1-\rho_3)\rho_4 > 1 \\
&\rho_1(1-(1-\rho_3)\rho_4) > \rho_1 \\
&\rho_1(1-(1-\rho_3)\rho_4)-1 > \rho_1-1
\end{aligned}
$$

Because $\rho_1 > 1$ the numerator of θ_2 is always positive.

Case 5 : $\rho_{1,2,3} > 1, \rho_4 < 1$

It is said that the numerators of θ_2 and θ_4 are both positive. First, we prove it for θ_2.

$$
\begin{aligned}
&1-\rho_3 < 0 \\
&(1-\rho_3)\rho_4 < 0 \\
&1-(1-\rho_3)\rho_4 > 1 \\
&\rho_1(1-(1-\rho_3)\rho_4) > \rho_1 \\
&\rho_1(1-(1-\rho_3)\rho_4)-1 > \rho_1-1
\end{aligned}
$$

A similar approach can be used for θ_4

$$
\begin{aligned}
&1-\rho_1 < 0 \\
&(1-\rho_1)\rho_2 < 0 \\
&1-(1-\rho_1)\rho_2 > 1 \\
&\rho_3(1-(1-\rho_1)\rho_2) > \rho_3 \\
&\rho_3(1-(1-\rho_1)\rho_2)-1 > \rho_3-1
\end{aligned}
$$

The numerators of θ_1 and θ_3 can be both positive or negative. First, this is proven for θ_1.

$$\begin{aligned}
&1-\rho_2 < 0\\
&(1-\rho_2)\rho_3 < 0\\
&1-(1-\rho_2)\rho_3 > 1\\
&\rho_4(1-(1-\rho_2)\rho_3) > \rho_4\\
&\rho_4(1-(1-\rho_2)\rho_3) - 1 > \rho_4 - 1
\end{aligned}$$

which can be positive or negative as $\rho_4 < 1$. The proof for θ_3 can be found below.

$$\begin{aligned}
&0 < 1-\rho_4 < 1\\
&0 < (1-\rho_4)\rho_1 < \rho_1\\
&1 > 1-(1-\rho_4)\rho_1 > 1-\rho_1\\
&\rho_2 > \rho_2(1-(1-\rho_4)\rho_1) > \rho_2(1-\rho_1)\\
&\rho_2 - 1 > \rho_2(1-(1-\rho_4)\rho_1) > \rho_2(1-\rho_1) - 1
\end{aligned}$$

Once again, with the use of two correct requirements for θ_1 and θ_3, all θ_i are positive.

Case 6 : $\rho_i > 1$

There are no additional requirements needed for this case. The proof for all θ_i is the same, therefore only $i = 1$ is done. The numerator is always positive:

$$\begin{aligned}
&1-\rho_2 < 0\\
&(1-\rho_2)\rho_3 < 0\\
&1-(1-\rho_2)\rho_3 > 1\\
&\rho_4(1-(1-\rho_2)\rho_3) > \rho_4\\
&\rho_4(1-(1-\rho_2)\rho_3) - 1 > \rho_4 - 1
\end{aligned}$$

and the denominator is also always positive.

$$\begin{aligned}
&\rho_1\rho_2\rho_3\rho_4 > 1\\
&\rho_1\rho_2\rho_3\rho_4 - 1 > 0
\end{aligned}$$

This proves the system behaves in a stable manner if an equilibrium point is reached.

Appendix F Proof for the partially asymptotically stable system

This appendix derives the conclusions presented in subsection 3.3.1. There are five routes that lead to the mode in which all queues are empty. Each route is analyzed by looking at one transition at a time and combining these transitions to give the restrictions on Y_i that make sure the system is asymptotically stable. Y_i^j means the time needed to empty queue i in mode j. The ODEs can be found in Appendix C.

Route 1 : $16 \to 8 \to 6$

For $16 \to 8$ to occur, we have to make sure that $Y_1^{16} < Y_2^{16}, Y_3^{16}, Y_4^{16}$, the queues in mode 8 are described by

$$\begin{aligned} Y_2^8 &= Y_2^{16} - Y_1^{16} \\ Y_3^8 &= Y_3^{16} - Y_1^{16} \\ Y_4^8 &= Y_4^{16} - Y_1^{16} \end{aligned}$$

The second transition, $8 \to 6$, has the condition $Y_3^8 < Y_4^8$, the third queue has to empty before the fourth. The second queue increases as can be seen in the ODEs. This leaves only the fourth queue, which has the value

$$\begin{aligned} Y_4^6 &= Y_4^8 - Y_3^8 \\ &= Y_4^{16} - Y_3^{16} - 2Y_1^{16} \end{aligned}$$

This gives the total restriction as

$$\left\{ (Y_1^{16}, Y_2^{16}, Y_3^{16}, Y_4^{16}) \middle| Y_1^{16} < Y_2^{16} \wedge Y_1^{16} < Y_3^{16} < Y_4^{16} \right\}$$

Route 2 : $16 \to 12 \to 8 \to 6$

For $16 \to 8$ to occur, we have to make sure that $Y_2^{16} < Y_1^{16}, Y_3^{16}, Y_4^{16}$, the queues in mode 12 are described by

$$\begin{aligned} Y_1^{12} &= Y_1^{16} - Y_2^{16} \\ Y_3^{12} &= Y_3^{16} - Y_2^{16} \\ Y_4^{12} &= Y_4^{16} - Y_2^{16} \end{aligned}$$

The second transition, $12 \rightarrow 8$, has the condition $Y_1^{12} < \frac{1}{1-\rho_3} Y_3^{12}, Y_4^{12}$, the first queue has to empty before the third and fourth. This condition can be written in terms of the values in mode 16, by substituting the above expressions

$$Y_1^{16} < \frac{1}{1-\rho_3}\left[Y_3^{16} - Y_2^{16}(1-(1-\rho_3))\right], Y_4^{16}$$

After the transition takes place, the queues are described by

$$\begin{aligned} Y_3^8 &= Y_3^{12} - (1-\rho_3)Y_1^{12} \\ Y_4^8 &= Y_4^{12} - Y_1^{12} \end{aligned}$$

For the last transition, from $8 \rightarrow 6$ the third queue has to empty before the fourth, $Y_3^8 < Y_4^8$. This restriction can also be rewritten to the terms in mode 16.

$$\begin{aligned} & Y_3^{12} - (1-\rho_3)Y_1^{12} < Y_4^{12} - Y_1^{12} \\ & Y_3^{12} + \rho_3 Y_1^{12} < Y_4^{12} \\ & Y_3^{16} - Y_2^{16} + \rho_3(Y_1^{16} - Y_2^{16}) < Y_4^{16} - Y_2^{16} \\ & Y_3^{16} + \rho_3(Y_1^{16} - Y_2^{16}) < Y_4^{16} \end{aligned}$$

With the knowledge that $Y_3^8 < Y_3^{12}$ (and by that leaving $\frac{1}{1-\rho_3}\left[Y_3^{16} - Y_2^{16}(1-(1-\rho_3))\right]$ out), because the ODEs are negative, this gives the total restriction as

$$\left\{(Y_1^{16}, Y_2^{16}, Y_3^{16}, Y_4^{16}) \middle| Y_2^{16} < Y_1^{16} < Y_3^{16} + \rho_3(Y_1^{16} - Y_2^{16}) < Y_4^{16}\right\}$$

Route 3 : $16 \rightarrow 12 \rightarrow 10 \rightarrow 6$

The first transition, $16 \rightarrow 12$, can be seen above in route 2. For $12 \rightarrow 10$, the condition $\frac{1}{1-\rho_3} Y_3^{12} < Y_1^{12}, Y_4^{12}$ has to be fulfilled. This requirement can be rewritten to the variables in mode 16.

$$\begin{aligned} & \frac{1}{1-\rho_3}(Y_3^{16} - Y_2^{16}) < Y_1^{16} - Y_2^{16}, Y_4^{16} - Y_2^{16} \\ & Y_3^{16} - Y_2^{16} < (1-\rho_3)(Y_1^{16} - Y_2^{16}), (1-\rho_3)(Y_4^{16} - Y_2^{16}) \\ & Y_3^{16} < (1-\rho_3)Y_1^{16} + \rho_3 Y_2^{16}, (1-\rho_3)Y_4^{16} + \rho_3 Y_2^{16} \end{aligned}$$

The queues in mode 10 are as follows

$$Y_1^{10} = Y_1^{12} - \frac{1}{1-\rho_3} Y_3^{12}$$
$$Y_4^{10} = Y_4^{12} - \frac{1}{1-\rho_3} Y_3^{12}$$

For $10 \to 6$ to take place, the condition $Y_1^{10} < \frac{1}{1-(1-\rho_3)\rho_4} Y_4^{10}$ has to be met, this can be rewritten

$$Y_1^{12} - \frac{1}{1-\rho_3} Y_3^{12} < \frac{1}{1-(1-\rho_3)\rho_4} (Y_4^{12} - \frac{1}{1-\rho_3} Y_3^{12})$$
$$Y_1^{12} < \frac{1}{1-(1-\rho_3)\rho_4} \left[Y_4^{12} - \frac{(1-\rho_3)\rho_4}{1-\rho_3} Y_3^{12} \right]$$
$$Y_1^{16} - Y_2^{16} < \frac{1}{1-(1-\rho_3)\rho_4} \left[(Y_4^{16} - Y_2^{16}) - \frac{(1-\rho_3)\rho_4}{1-\rho_3} (Y_3^{16} - Y_2^{16}) \right]$$
$$Y_1^{16} < \frac{1}{1-(1-\rho_3)\rho_4} (Y_4^{16} - Y_2^{16}) - \frac{\rho_4}{1-(1-\rho_3)\rho_4} (Y_3^{16} - Y_2^{16}) + Y_2^{16}$$
$$Y_1^{16} < \frac{1}{1-(1-\rho_3)\rho_4} \left[Y_4^{16} + \rho_4 Y_3^{16} + \rho_3 \rho_4 Y_2^{16} \right]$$

This gives the total restriction as

$$\left\{ (Y_1^{16}, Y_2^{16}, Y_3^{16}, Y_4^{16}) \middle| Y_2^{16} < Y_3^{16} < Y_1^{16} < \frac{1}{1-(1-\rho_3)\rho_4} \left[Y_4^{16} + \rho_4 Y_3^{16} + \rho_3 \rho_4 Y_2^{16} \right] \right\}$$

Route 4 : $16 \to 14 \to 10 \to 6$

The first transition is from $16 \to 14$ with the condition $Y_3^{16} < Y_1^{16}, Y_2^{16}, Y_4^{16}$

$$Y_1^{14} = Y_1^{16} - Y_3^{16}$$
$$Y_2^{14} = Y_2^{16} - Y_3^{16}$$
$$Y_4^{14} = Y_4^{16} - Y_3^{16}$$

The next transition is from $14 \to 10$, the condition $Y_2^{14} < Y_1^{14}, \frac{1}{1-\rho_4} Y_4^{14}$ has to be met, or, in terms of mode 16.

$$Y_2^{16} - Y_3^{16} < Y_1^{16} - Y_3^{16}, \frac{1}{1-\rho_4}(Y_4^{16} - Y_3^{16})$$

$$Y_2^{16} < Y_1^{16}, \frac{1}{1-\rho_4}(Y_4^{16} - \rho_4 Y_3^{16})$$

The queues in mode 10 are described by

$$Y_1^{10} = Y_1^{14} - Y_2^{14}$$
$$Y_4^{10} = Y_4^{14} - (1-\rho_4)Y_2^{14}$$

For the last transition $10 \to 6$ to take place, the condition $Y_1^{10} < \frac{1}{1-(1-\rho_3)\rho_4} Y_4^{10}$ has to be met. This condition can be rewritten in terms of mode 16.

$$Y_1^{14} - Y_2^{14} < \frac{1}{1-(1-\rho_3)\rho_4}(Y_4^{14} - (1-\rho_4)Y_2^{14})$$

$$Y_1^{14} < \frac{1}{1-(1-\rho_3)\rho_4}\left[Y_4^{14} - \left[(1-\rho_4) - 1 + (1-\rho_3)\rho_4\right]Y_2^{14}\right]$$

$$Y_1^{14} < \frac{1}{1-(1-\rho_3)\rho_4}\left[Y_4^{14} - \rho_3\rho_4 Y_2^{14}\right]$$

$$Y_1^{16} - Y_3^{16} < \frac{1}{1-(1-\rho_3)\rho_4}\left[(Y_4^{16} - Y_3^{16}) - \rho_3\rho_4(Y_2^{16} - Y_3^{16})\right]$$

$$Y_1^{16} < \frac{1}{1-(1-\rho_3)\rho_4}\left[Y_4^{16} - \rho_3\rho_4 Y_2^{16} + (2\rho_3 - 1)\rho_4)Y_3^{16}\right]$$

This gives the total restriction as

$$\left\{(Y_1^{16}, Y_2^{16}, Y_3^{16}, Y_4^{16}) \middle| Y_3^{16} < Y_2^{16} < Y_1^{16} < \frac{1}{1-(1-\rho_3)\rho_4}\left[Y_4^{16} - \rho_3\rho_4 Y_2^{16} + (2\rho_3 - 1)\rho_4)Y_3^{16}\right]\right\}$$

Route 5 : $16 \to 14 \to 6$

The last route starts with the transition $16 \to 14$ which can be seen above in route 4. The second transition is $14 \to 6$ with the condition $Y_1^{14} < Y_2^{14}, \frac{1}{1-\rho_4} Y_4^{14}$, or, in terms of the variables in mode 16

$$Y_1^{16} - Y_3^{16} < Y_2^{16} - Y_3^{16}, \frac{1}{1-\rho_4}(Y_4^{16} - Y_3^{16})$$

$$Y_1^{16} < Y_2^{16}, \frac{1}{1-\rho_4}(Y_4^{16} - \rho_4 Y_3^{16})$$

The total restriction is as follows

$$\left\{ (Y_1^{16}, Y_2^{16}, Y_3^{16}, Y_4^{16}) \middle| Y_3^{16} < Y_1^{16} < Y_2^{16} \wedge Y_1^{16} < \frac{1}{1-\rho_4}(Y_4^{16} - \rho_4 Y_3^{16}) \right\}$$

Appendix G Proof for the transient cyclic behaviour

In this appendix the analysis of the transient cyclic behaviour (subsection 3.3.2) is presented in full. The cyclic behaviour is witnessed between modes $15 \rightarrow 14 \rightarrow 12 \rightarrow 8 \rightarrow 15 \rightarrow \ldots$. The analysis is done in steps, each time examining one transition. Y_i^j means the time needed to empty queue i in mode j. The ODEs can be found in Appendix C.

The analysis starts with the system entering mode 15 from mode 8 for the k-th time, for readability, the index k is left out. Upon entering mode 15 from mode 8, only the second and third queue are non-empty. For the first transition, $15 \rightarrow 14$, the condition $Y_3^{15} < Y_2^{15}$ has to hold. The queues can be described by

$$\begin{aligned} Y_1^{14} &= (\rho_1 - 1)Y_3^{15} \\ Y_2^{14} &= Y_2^{15} - Y_3^{15} \end{aligned}$$

For the second transition $14 \rightarrow 12$, the condition $Y_2^{14} < Y_1^{14}$ has to hold, this can be rewritten in terms of the variables in mode 15

$$\begin{aligned} Y_2^{15} - Y_3^{15} &< (\rho_1 - 1)Y_3^{15} \\ Y_2^{15} &< \rho_1 Y_3^{15} \end{aligned}$$

Upon entering, the queues in mode 12 are as follows

$$\begin{aligned} Y_1^{12} &= Y_1^{14} - Y_2^{14} \\ Y_4^{12} &= (\rho_4 - 1)Y_2^{14} \end{aligned}$$

For $12 \rightarrow 8$, $Y_1^{12} < Y_4^{12}$ has to hold, the restriction is rewritten in terms of the variables in mode 15

$$\begin{aligned} Y_1^{14} - Y_2^{14} &< (\rho_4 - 1)Y_2^{14} \\ Y_1^{14} &< \rho_4 Y_2^{14} \\ (\rho_1 - 1)Y_3^{15} &< \rho_4 (Y_2^{15} - Y_3^{15}) \\ \left[\frac{\rho_1 - 1}{\rho_4} + 1\right] Y_3^{15} &< Y_2^{15} \end{aligned}$$

At the start of mode 8, the queues have these values

$$Y_3^8 = (\rho_3 - 1)Y_1^{12}$$
$$Y_4^8 = Y_4^{12} - Y_1^{12}$$

For the last transition, $8 \rightarrow 15$, the condition $Y_4^8 < Y_3^8$ has to be fulfilled. This condition can be rewritten as

$$Y_4^{12} - Y_1^{12} < (\rho_3 - 1)Y_1^{12}$$
$$Y_4^{12} < \rho_3 Y_1^{12}$$
$$(\rho_4 - 1)Y_2^{14} < \rho_3 (Y_1^{14} - Y_2^{14})$$
$$\frac{(\rho_4 - 1)}{\rho_3} Y_2^{14} < Y_1^{14} - Y_2^{14}$$
$$\left[\frac{(\rho_4 - 1) + \rho_3}{\rho_3}\right] Y_2^{14} < Y_1^{14}$$
$$\left[\frac{(\rho_4 - 1) + \rho_3}{\rho_3}\right] (Y_2^{15} - Y_3^{15}) < (\rho_1 - 1)Y_3^{15}$$
$$Y_2^{15} < \left[\frac{(\rho_1 - 1)\rho_3}{(\rho_4 - 1) + \rho_3} + 1\right] Y_3^{15}$$

which can be rewritten to

$$Y_2^{15} < \left[\rho_1 - \frac{(\rho_1 - 1)(\rho_4 - 1)}{\rho_3 + \rho_4 - 1}\right] Y_3^{15}$$

This finalizes the cycle, by combining all restrictions, one total restriction is derived

$$\left[\frac{\rho_1 - 1}{\rho_4} + 1\right] Y_3^{15} < Y_2^{15} < \left[\rho_1 - \frac{(\rho_1 - 1)(\rho_4 - 1)}{\rho_3 + \rho_4 - 1}\right] Y_3^{15}$$

When returning to mode 15 for the $(k+1)$-th time, the queues can be described by

$$Y_2^{15}(k+1) = (\rho_2 - 1)Y_4^8$$
$$Y_3^{15}(k+1) = Y_3^8 - Y_4^8$$

which can be written in terms of the variables in mode 15, for the second queue

$$
\begin{aligned}
Y_2^{15}(k+1) &= (\rho_2 - 1)Y_4^8 \\
&= (\rho_2 - 1)(Y_4^{12} - Y_1^{12}) \\
&= (\rho_2 - 1)((\rho_4 - 1)Y_2^{14} - (Y_1^{14} - Y_2^{14})) \\
&= \rho_4(\rho_2 - 1)Y_2^{14} - (\rho_2 - 1)Y_1^{14} \\
&= \rho_4(\rho_2 - 1)(Y_2^{15} - Y_3^{15}) - (\rho_2 - 1)(\rho_1 - 1)Y_3^{15} \\
&= \rho_4(\rho_2 - 1)Y_2^{15} - (\rho_2 - 1)(\rho_1 + \rho_4 - 1)Y_3^{15}
\end{aligned}
$$

and for the third queue

$$
\begin{aligned}
Y_3^{15}(k+1) &= Y_3^8 - Y_4^8 \\
&= (\rho_3 - 1)Y_1^{12} - (Y_4^{12} - Y_1^{12}) \\
&= \rho_3 Y_1^{12} - Y_4^{12} \\
&= \rho_3(Y_1^{14} - Y_2^{14}) - (\rho_4 - 1)Y_2^{14} \\
&= \rho_3 Y_1^{14} - (\rho_4 + \rho_3 - 1)Y_2^{14} \\
&= \rho_3(\rho_1 - 1)Y_3^{15} - (\rho_4 + \rho_3 - 1)(Y_2^{15} - Y_3^{15}) \\
&= \left[\rho_3(\rho_1 - 1) + (\rho_4 + \rho_3 - 1)\right]Y_3^{15} - (\rho_4 + \rho_3 - 1)Y_2^{15} \\
&= \left[\rho_3\rho_1 + \rho_4 - 1\right]Y_3^{15} - (\rho_4 + \rho_3 - 1)Y_2^{15}
\end{aligned}
$$

This can be rewritten in the form $\underline{Y}(k+1) = A(\rho)\underline{Y}(k)$, or in matrix-vector notation

$$
\begin{pmatrix} Y_2^{15}(k+1) \\ Y_3^{15}(k+1) \end{pmatrix} = \begin{pmatrix} \rho_4(\rho_2 - 1) & -(\rho_2 - 1)(\rho_1 + \rho_4 - 1) \\ -(\rho_4 + \rho_3 - 1) & \rho_3\rho_1 + \rho_4 - 1 \end{pmatrix} \begin{pmatrix} Y_2^{15}(k) \\ Y_3^{15}(k) \end{pmatrix}
$$